Homework II

Due Jan 26, 2018.

1 For an incompressible flow, show that

$$D = \int_V \boldsymbol{u} \cdot \nabla^2 \boldsymbol{u} \, \mathrm{d}V = \int_S W \, \mathrm{d}S - 2 \int_V e_{ij} e_{ij} \, \mathrm{d}V,$$

where $e_{ij} = (\partial u_i / \partial x_j + \partial u_j / \partial x_i)/2$ is the strain rate and you are to determine W. If the stress tensor is given by $\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij}$, show the surface term vanishes if $f_i u_i = 0$. In that case, what is the sign of D?

2 Consider the velocity field

$$\boldsymbol{u} = \left(-y, x, \frac{1}{x^2 + y^2 + t + 1} \right).$$

At $t = 2\pi$, compute the streamlines and the streakline made up of dye released from $(1, 0, 0)$ during $0 \leq t \leq 2\pi$. Compute the particle path starting at $(1, 0, 0)$ at $t = 0$. [Write the results in parametric form.]

3 Compute the shear e_{ij} and vorticity for the following flows

$$(-xz, -yz, z^2), \qquad (-y, x, 0)$$

Show that these flows are incompressible. Do the same for the flow

$$(A \cos ax \sin by \sin cz, B \sin ax \cos by \sin cz, C \sin ax \sin by \cos cz),$$

obtaining the condition for the flow to be incompressible. (This is called the Taylor–Green vortex.)

4 The flow known as Hill's spherical vortex is given (in cylindrical polar coordinates) inside the sphere of radius a by the Stokes streamfunction

$$\psi = \frac{1}{10} A \sigma^2 (a^2 - z^2 - \sigma^2).$$

What is the vorticity inside the sphere? Verify that the velocity potential outside the sphere $\phi = U \cos\theta (r + a^3/2r^2)$ (in spherical polar coordinates) satisfies Laplace's equation. Show that requiring that the velocity field be continuous at the surface of the sphere leads to $U = (2/15)a^2 A$. You will probably want to use

$$\boldsymbol{x} = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta) = (\sigma \cos\phi, \sigma \sin\phi, z).$$

to obtain

$$\hat{\theta} = \cos\theta \, \hat{\sigma} - \sin\theta \, \hat{\mathbf{z}}.$$