Midterm Solution

1 Particle path: first solve

$$\frac{dx}{dt} = \cos t, \qquad \frac{dz}{dt} = -z.$$

This gives $x(t) = \sin t$ and $z(t) = e^{-t}$ using the initial condition. Now solve

$$\frac{dy}{dt} = x = \sin t,$$

giving $y(t) = 1 - \cos t$. Streamlines at $t = 0$: solve

$$\frac{dx}{1} = \frac{dy}{x} = -\frac{dz}{z}.$$

The coordinate x can be used to parameterize the streamlines, giving $y = x^2/2 + A$, $z = Be^{-x}$. Streaklines: solve the particle path equations with initial position $(0, 0, 1)$ at $t = t_*$, giving

$$x(t) = \sin t - \sin t_*, \qquad y(t) = -\cos t - t \sin t_* + \cos t_* + t_* \sin t_*, \quad z(t) = e^{-t+t_*}.$$

At $t = \pi$, these give $(-\sin t_*, 1 - \pi \sin t_* + \cos t_* + t_* \sin t_*, e^{-\pi+t_*})$ with $0 \le t \le \pi$.

2 (i) The vorticity is in the z-direction and is given by $\omega = 2\Omega$ for $r < a$ and $\omega = 0$ for $r > a$.
(ii) The circulation is the area integral of vorticity. The vorticity is piecewise constant. For $R < a$, the circulation is the area of the circle of radius R times 2Ω, so $\Gamma = 2\Omega\pi R^2$. For $R > a$, there is no vorticity outside the circle of radius a, so the circulation is the area of the circle of radius a times 2Ω, so $\Gamma = 2\Omega\pi a^2$.
(iii) There is no flow in the vertical direction, so

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g,$$

which gives $p = -\rho g z + A(r)$. The radial equation gives

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} = -\frac{u_\theta^2}{r} = \begin{cases} -\Omega^2 r & \text{for } 0 < r < a, \\ -\Omega^2 a^4/r^3 & \text{for } a < r. \end{cases}$$

This gives

$$p = \begin{cases} \Omega^2 r^2/2 + B(z) & \text{for } 0 < r < a, \\ \Omega^2 a^4/2r^2 + B(z) & \text{for } a < r. \end{cases}$$

Hence

$$p = P_0 - \rho g z + \begin{cases} \Omega^2 r^2/2 & \text{for } 0 < r < a, \\ \Omega^2 a^4/2r^2 & \text{for } a < r. \end{cases}$$

The curves of constant p are given by $z = A + Br^2$ (parabolas) for $r < a$ and by $z = C + Dr^{-2}$ for $r > a$, where A, B, C and D can be found from the previous expression.

3 (i) The steady and advective terms vanish. The y- and z-components of velocity vanish so p does not depend on y and z. The x-component gives

$$0 = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} + \nu\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}z} + 2A\nu.$$

(ii)

$$Q = \int_{-h}^{h} A(y^2 - h^2)\,\mathrm{d}y = -\frac{4Ah^3}{3}.$$

Hence $A = -3Q/4h^3 < 0$.
(ii) From (i) the pressure gradient is constant: $\mathrm{d}p/\mathrm{d}x = 2A\rho\nu = 2\mu A = -3\mu/2h^3$. The rate of strain e_{ij} and stress $\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}$ tensors are

$$e_{ij} = \begin{pmatrix} 0 & Ay & 0 \\ Ay & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \sigma_{ij} = \begin{pmatrix} -2\mu Ax & 2\mu Ay & 0 \\ 2\mu Ay & -2\mu Ax & 0 \\ 0 & 0 & -2\mu Ax \end{pmatrix},$$

using $p = -2 - \nu Ax$ (i.e. setting the pressure to vanish at $x = 0$).
(iii) The shear stress is $\mu \mathrm{d}u/\mathrm{d}y = 2\mu Ay$. At the boundaries, this is $2\mu Ah$, a negative constant. The force on the wall is minus the integral over area (per unit width), i.e. $F = -2\mu AhL = 3\mu QL/2h^2$ toward the right since $Q > 0$ (the fluid tries to drag the plate along with it).