Homework 0

1 Take dot product and cross product with $\boldsymbol{u}$:

$$\boldsymbol{u}\cdot\boldsymbol{x} = \boldsymbol{u}\cdot\boldsymbol{v}; \qquad u^2\boldsymbol{x} - (\boldsymbol{u}\cdot\boldsymbol{x})\boldsymbol{u} + \boldsymbol{u}\times\boldsymbol{x} = \boldsymbol{u}\times\boldsymbol{v}.$$

where $u = |\boldsymbol{u}|$. Substitute $\boldsymbol{u}\cdot\boldsymbol{x}$ and $\boldsymbol{x}\times\boldsymbol{u}$ into the final equation:

$$u^2\boldsymbol{x} - (\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{u} + \boldsymbol{x} - \boldsymbol{v} = \boldsymbol{u}\times\boldsymbol{v}.$$

Now solve for $\boldsymbol{x}$:

$$\boldsymbol{x} = (1+u^2)^{-1}[(\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{u}\times\boldsymbol{v}].$$

2 We have $\boldsymbol{\nabla} f = (6x-2y-3, 2y-2x)$. This vanishes at $x = y = 3/4$ which is not inside the circle. Write $x = \cos\theta, y = \sin\theta$ on the boundary. Then $f(\theta) = 3\cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta - 3\cos\theta$. This is a function of one variable, and its maxima and minima can be found by differentiating. The stationarity condition is

$$f'(\theta) = -4\cos\theta\sin\theta + 4\sin\theta^2 + 3\sin\theta - 2 = 0.$$

The best way to solve this is to square it so that it becomes the polynomial equation

$$32\sin^4\theta + 24\sin^3\theta - 23\sin^2\theta - 12\sin\theta + 4 = 0.$$

This polynomial has four roots, and the relevant one, as shown in Figure 1a, is $\sin\theta = 0.6928$, which corresponds to the point on the boundary $(0.7211, 0.6928)$. Next $\boldsymbol{\nabla} g = (6x-4y-3, 2y-4x)$. Again the extremeum is outside the circle. Now $g(\theta) = 3\cos^2\theta + \sin^2\theta - 4\sin\theta\cos\theta - 3\cos\theta$. The stationarity condition is

$$g'(\theta) = -4\cos\theta\sin\theta + 8\sin\theta^2 + 3\sin\theta - 4 = 0.$$

Hence

$$80\sin^4\theta + 48\sin^3\theta - 71\sin^2\theta - 24\sin\theta + 16 = 0.$$

This polynomial also has four roots, and the relevant one, as shown in Figure 1b, is $\sin\theta = 0.6985$, which corresponds to the point on the boundary $(0.7156, 0.6985)$.

3 Use suffix notation.

- Remember that $\partial r/\partial x_i = x_i/r$ (which can be obtained by taking the gradient of $r^2 = x_j x_j$).

$$\nabla^2 r^n = \frac{\partial^2}{\partial x_i \partial x_i} r^n = \frac{\partial}{\partial x_i} n x_i r^{n-2} = n[(n-2) + \delta_{ii}]r^{n-2}.$$

  So in three dimensions, the answer is $n(n+1)r^{n-2}$, in two dimensions $n^2 r^{n-2}$.

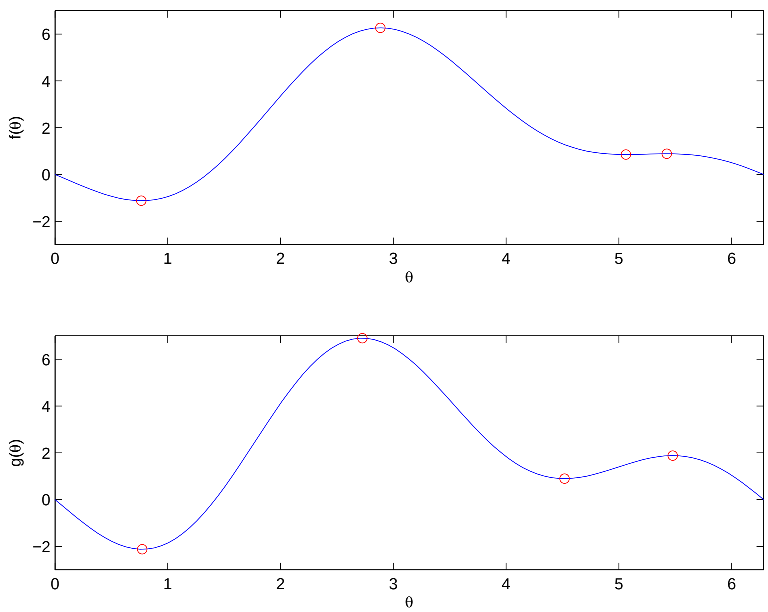

Figure 1: $f(\theta)$ and $g(\theta)$ with maximum and minima.

- This uses the result $\partial x_i/\partial x_j = \delta_{ij}$.

$$[\nabla \times (\Omega \times \boldsymbol{x})]_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}\epsilon_{klm}\Omega_l x_m = \Omega_l(\delta il\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial x_m}{\partial x_j} = 2\Omega_i.$$

  This is just 2Ω in vector notation.

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$$(\nabla f)_k = A_{kj}x_j + A_{jk}x_j + B_k = (A_{jk} + A_{kj})x_j + B_k.$$

  If A is symmetric, the first term is just $2A_{kj}x_j$.

4 Vector divergence theorem: the integral is

$$-\int_V \boldsymbol{\nabla} p \, \mathrm{d}V = \rho g \boldsymbol{k} \int_V \mathrm{d}V = \rho g V,$$

where V is the volume surrounded by the surface. If p is the pressure, this shows that the force on a submerged body in a fluid at rest is equal to the weight of fluid displaced by the body.

5 Let S be a surface far from the origin (which we take later to tend to infinity). Then

$$\int \omega_i(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \int \frac{\partial}{\partial x_j}(x_i\omega_j) \, \mathrm{d}\boldsymbol{x} = \int_S x_i\omega_j \, \mathrm{d}S_j = 0,$$

since $\partial\omega_j/\partial x_j = 0$ and the surface integral vanishes since ω is zero outside a bounded region. For the second integral (call it $\boldsymbol{D}$),

$$D_i = \int \frac{\partial}{\partial x'_k}(x_j x'_j x'_i \omega'_k)\,\mathrm{d}\boldsymbol{x}' - \int x_j x'_i \omega'_j\,\mathrm{d}\boldsymbol{x}' = \int_S x_j x'_j x'_i \omega'_k\,\mathrm{d}S_k - \int x_j x'_i \omega'_j\,\mathrm{d}\boldsymbol{x}.$$

Once again the surface integral vanishes. Now add this expression to $\boldsymbol{D}$ and divide by two:

$$\boldsymbol{D} = \frac{1}{2}\int ([(\boldsymbol{x}\cdot\boldsymbol{x}')\omega' + (\boldsymbol{x}\cdot\omega')\boldsymbol{x}']\,\mathrm{d}\boldsymbol{x}' = \frac{1}{2}\int [\boldsymbol{x}\times(\omega'\times\boldsymbol{x}')]\,\mathrm{d}\boldsymbol{x}' = \frac{1}{2}\boldsymbol{x}\times\int \omega'\times\boldsymbol{x}'\,\mathrm{d}\boldsymbol{x}'.$$

The $\boldsymbol{x}$ can be taken out of the last integral since it is independent of $\boldsymbol{x}'$. The first result shows that the integral of a solenoidal vector field over all space is zero. (The average is zero too, since one would take the above and divide by the volume of the sphere inside S.)

6 Call the integral I_{ijkl}. It is isotropic since if one transforms to another set of axes, the sphere does not change. Hence it must be equal to the isotropic fourth-rank tensor given in the question. Consider the contraction $i = j$, $k = l$:

$$I = I_{iikk} = \int_V x_i x_i x_k x_k\,\mathrm{d}V = \int r^4\,\mathrm{d}V = 9\lambda + 3\mu + 3\nu.$$

Two other similar contractions give $I = 3\lambda + 9\mu + 3\nu = 3\lambda + 3\mu + 9\nu$. Hence by symmetry, $\lambda = \mu = \nu$ and we can calculate I:

$$I = \int_0^a r^4(4\pi r^2)\,\mathrm{d}r = \frac{4\pi a^7}{7} = 15\lambda.$$

The final result is

$$\int_V x_i x_i x_k x_k\,\mathrm{d}V = \frac{4\pi a^7}{105}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

7 At $(3,0,4)$, $r = 5$ and the tensor becomes in matrix form

$$\begin{pmatrix} 4 & 0 & 12 \\ 0 & -5 & 0 \\ 12 & - & 11 \end{pmatrix}.$$

The characteristic equation for this matrix is $\lambda^3 - 10\lambda^2 - 175\lambda - 500 = 0$, with solutions -5, -5 and 20. The corresponding orthonormal eigenvectors are $(0,1,0)$, $(4/5,0,-3/5)$ and $(3/5,0,4/5)$.