Solutions I

1 Convert the vector expression to suffices:

$$\begin{aligned} t_i &= -pn_i + \mu\left(2n_j\frac{\partial u_i}{\partial x_j} + \epsilon_{ijk}n_j\epsilon_{klm}\frac{\partial u_m}{\partial x_l}\right) \\ &= -pn_i + \mu\left(2n_j\frac{\partial u_i}{\partial x_j} + (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})n_j\frac{\partial u_m}{\partial x_l}\right) \\ &= -pn_i + \mu\left(2n_j\frac{\partial u_i}{\partial x_j} + n_j\frac{\partial u_j}{\partial x_i} - n_j\frac{\partial u_i}{\partial x_j}\right). \end{aligned}$$

2 The components of τ_{ij} are

$$\tau_{ij} = \begin{pmatrix} 0 & -2Uy/a^2 & -2Uz/b^2 \\ -2Uy/a^2 & 0 & 0 \\ -2Uz/b^2 & 0 & 0 \end{pmatrix}.$$

The normal to the plane $x = 0$ has components $(1, 0, 0)$, so the matrix-vector product gives the shear stress $(0, -2Uy/a^2, -2Uz/b^2)$. The magnitude of the stress is hence $f = 2U(y^2/a^4 + z^2/b^4)^{1/2}$. This is smallest on the axis where $y = z = 0$ and largest on the boundary, where we write $y = a\cos\theta$, $z = b\cos\theta$. Then

$$\frac{f^2}{4U^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} = \frac{1}{a^2} + \left(\frac{1}{b^2} - \frac{1}{a^2}\right)\sin^2\theta = \frac{1}{b^2} + \left(\frac{1}{a^2} - \frac{1}{b^2}\right)\cos^2\theta.$$

If $a < b$ this is maximum when $\theta = 0$ and $\theta = \pi$. If $a > b$ this is maximum when $\theta = \pi/2$ and $\theta = 3\pi/2$. So the shear stress magnitude is largest at the intersection of the semi-minor axis and the boundary. If $a = b$ (circle), the shear stress has the same value over the whole boundary.

3 Along any circle $\boldsymbol{n} = \hat{\boldsymbol{r}}$ so

$$\boldsymbol{u}\cdot\boldsymbol{n} = \nabla\phi\cdot\hat{\boldsymbol{r}} = \frac{\partial\phi}{\partial r} = U\cos\theta\left(1 - \frac{a^2}{r^2}\right).$$

This vanishes along the circle $r = a$. The function p is given along $r = a$ by

$$-\tfrac{1}{2}\rho\left(4U^2\sin^2\theta + \frac{\Gamma^2}{4\pi^2a^2} - \frac{2\Gamma U}{\pi a}\sin\theta\right).$$

Along the circle we have $\mathrm{d}\boldsymbol{S} = a(\cos\theta, \sin\theta)\,\mathrm{d}\theta$. Hence

$$-\int_S p\,\mathrm{d}\boldsymbol{S} = \tfrac{1}{2}\rho\int_0^{2\pi}\left(4U^2\sin^2\theta + \frac{\Gamma^2}{4\pi^2a^2} - \frac{2\Gamma U}{\pi a}\sin\theta\right)(\cos\theta, \sin\theta)a\,\mathrm{d}\theta.$$

This gives $-\rho\Gamma U\boldsymbol{j}$. Along any circle $\boldsymbol{x} = \boldsymbol{n}$ so $\boldsymbol{x}\times\boldsymbol{n} = \mathbf{0}$. Therefore the last integral vanishes.

4 The equation for the streamlines is $\mathrm{d}x/u = \mathrm{d}y/v$, which leads to

$$\frac{1+t}{x}\,\mathrm{d}x = \frac{2+t}{2y}\,\mathrm{d}y \qquad \text{i.e.} \quad |x|^{1+t} = A|y|^{1+t/2}.$$

Along the pathlines

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x}{1+t}, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{2y}{2+t}.$$

Using the initial condition, this gives in parametric form

$$x = 1+t, \qquad y = \left(\frac{2+t}{2}\right)^2$$

for $t \geq 0$. Solving for $y(x)$ gives

$$y = \left(\frac{1+x}{2}\right)^2$$

for $x \geq 1$. Write t_* for the time at which the particle passes through $(1,1)$. Then at $t = 1$

$$x = \frac{2}{1+t*}, \qquad y = \left(\frac{3}{2+t*}\right)^2$$

for $0 \leq t_* \leq 1$. Solving for $y(x)$ gives

$$y = \left(\frac{3x}{2+x}\right)^2$$

for $1 \leq x \leq 2$.