Solutions II

1 The flow is incompressible so

$$\frac{\partial e_{ij}}{\partial x_j} = \frac{1}{2}\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{2}\frac{\partial^2 u_j}{\partial x_j \partial x_i} = \frac{1}{2}\frac{\partial^2 u_i}{\partial x_j \partial x_j}.$$

Hence

$$D = \int_V u_i \frac{\partial^2 u_i}{x_j x_j}\,\mathrm{d}V = \int_V 2u_i \frac{\partial e_{ij}}{\partial x_j}\,\mathrm{d}V = 2\int_S e_{ij} u_i n_j\,\mathrm{d}S - 2\int e_{ij}\frac{\partial u_i}{\partial x_j}\,\mathrm{d}V,$$

so that $W = 2\int_S e_{ij} u_i n_j$ on the boundary. We have $u_i e_{ij} n_j = u_i(\tau_{ij} + p\delta_{ij})/(2\mu) n_j = u_i(f_i + pn_i)/(2\mu)$. On the boundary $u_i n_i = 0$ by the no-penetration condition. If $f_i u_i = 0$ as well, the surface integral vanishes. Then $D \leq 0$ because the remaining integrand is positive (D will vanish if all the e_{ij} terms are zero, i.e. if the velocity field is linear).

2 At $t = 2\pi$, the equations for the streamlines are

$$-\frac{\mathrm{d}x}{y} = \frac{\mathrm{d}y}{x} = (x^2 + y^2 + 2\pi + 1)\,\mathrm{d}z.$$

Parameterize with s. Then we need to solve

$$\frac{\mathrm{d}x}{\mathrm{d}s} = -y, \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = x, \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = \frac{1}{x^2 + y^2 + 2\pi + 1}.$$

The solution to the first two of these equations that passes through the point (x_0, y_0, z_0) is

$$x = x_0 \cos s - y_0 \sin s, \qquad y = x_0 \sin s + y_0 \cos s.$$

Hence $x^2 + y^2 = x_0^2 + y_0^2$ which is independent of s, and the solution to the final equation is

$$z = \frac{s}{x_0^2 + y_0^2 + 2\pi + 1} + z_0.$$

The equations for particle paths and streamlines are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{x^2 + y^2 + t + 1}.$$

The solution to the first two of these equations passing through $(1, 0, 0)$ at $t = t_*$ is

$$x = \cos(t - t_*), \qquad y = \sin(t - t_*).$$

Once again $x^2 + y^2 = 1$ is a constant. The final equation then has solution

$$z = \log \frac{t+2}{t_* + 2}.$$

The particle paths correspond to $t_* = 0$ so

$$x = \cos t, \qquad y = \sin t, \qquad z = \log(1 + t/2).$$

for $0 \le t \le 2\pi$. For streaklines, $t = 2\pi$ and

$$x = \cos t_*, \qquad y = -\sin t_*, \qquad z = \log \frac{2\pi + 2}{t_* + 2}.$$

where $0 \le t_* \le 2\pi$.

3 Strains:

$$e_{ij} = \begin{pmatrix} -z & 0 & -z/2 \\ 0 & -z & -y/2 \\ -z/2 & -y/2 & 2z \end{pmatrix}$$

and 0 for second flow, since it is solid-body rotation and the tensor $\partial u_i/\partial x_j$ is antisymmetric. The vorticities are $(y, -x, 0)$ and $(0, 0, 2)$. For the Taylor flow

$$e_{ij} = \begin{pmatrix} -aAs_as_bs_c & (bAc_ac_bs_c + aCs_as_bs_z)/2 & (cAc_as_bc_c + aCc_as_bc_c)/2 \\ (bAc_ac_bs_c + aBc_ac_bs_c)/2 & -bBs_as_bs_c & (bBs_as_bs_c + bCs_ac_bc_c)/2 \\ (cAc_as_bc_c + aCc_as_bc_c)/2 & (bBs_as_bs_c + bCs_ac_bc_c)/2 & -cCs_as_bs_c \end{pmatrix}$$

using a reduced notation, and the vorticity is

$$((bC - cB)s_ac_bc_c, (cA - aC)c_as_bc_c, (aB - bA)c_ac_bs_c)$$

The divergences of the first two velocity fields are zero (see the trace of e_{ij}). For the third, we compute the trace of e_{ij} and find the condition $aA + bB + cC = 0$.

4 The velocity inside the sphere is given by

$$\boldsymbol{u} = \frac{1}{\sigma}\frac{\partial \psi}{\partial z}\hat{\sigma} - \frac{1}{\sigma}\frac{\partial \psi}{\partial \sigma}\hat{\mathbf{z}} = \frac{1}{10}A[-2\sigma z\hat{\sigma} - (2a^2 - 2z^2 - 4\sigma^2)\hat{\mathbf{z}}].$$

The corresponding vorticity is

$$\left(\frac{\partial u_\sigma}{\partial z} - \frac{\partial u_z}{\partial \sigma}\right)\hat{\phi} = A\sigma\hat{\phi}.$$

The velocity outside the sphere is given by

$$\boldsymbol{u} = U\cos\theta\left(1 - \frac{a^2}{2r^3}\right)\hat{r} - U\sin\theta\left(1 + \frac{a^3}{2r^3}\right)\hat{\theta}.$$

The Laplacian of ϕ vanishes. Requiring the velocity to be continuous across the boundary gives

$$\frac{1}{10}A[-2\sigma z\hat{\sigma} + 2\sigma^2\hat{\mathbf{z}}] = -\frac{3}{2}U\sin\theta\hat{\theta}.$$

The boundary is given by $\sigma^2 + z^2 = a^2$ in cylindrical coordinates. On it the relations $\sigma = a\sin\theta$ and $z = a\cos\theta$ hold. The expression for $\boldsymbol{x}$ in spherical polar and cylindrical polar coordinates give

$$\hat{\theta} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), \qquad \hat{\sigma} = (\cos\phi, \sin\phi, 0), \qquad \hat{\mathbf{z}} = (0, 0, 1).$$

This gives the relation $\hat{\theta} = \cos\theta\,\hat{\sigma} - \sin\theta\,\hat{\mathbf{z}}$. Hence on the boundary

$$\frac{1}{10}Aa^2\sin\theta[-2\cos\theta\hat{\sigma} + 2\sin\theta\hat{\mathbf{z}}] = \frac{1}{10}Aa^2\sin\theta[-2\hat{\theta}] = -\frac{3}{2}U\sin\theta\hat{\theta}.$$

Hence $U = 2Aa^2/15$.