# Solutions III

**1** Compute the acceleration of a fluid particle for the velocity fields $\Omega(-y, x, 0)$ and $e(x, y, -2z)$. Discuss.
Both flows are incompressible. First

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \left(-\Omega y \frac{\partial}{\partial x} + \Omega x \frac{\partial}{\partial y}\right)(-\Omega y, \Omega x, 0) = \Omega^2(x, -y, 0);$$

this is solid body rotation and the acceleration points to the $z$-axis. Then

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \left(ex \frac{\partial}{\partial x} + ey \frac{\partial}{\partial y} - 2ez \frac{\partial}{\partial z}\right)(ex, ey, -2ez) = e^2(x, y, 4z);$$

this is a stagnation-like flow with inflow toward the $z$-axis and outflow along the $z$-axis. The acceleration is not related in any obvious way to the velocity or geometry.

**2**

$$\boldsymbol{\nabla} \cdot \boldsymbol{\omega} = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) = \frac{\partial}{\partial x_i} \epsilon_{ijk} \frac{\partial u_k}{\partial x_j} = \epsilon_{ijk} \frac{\partial^2}{\partial x_i \partial x_j} u_k = 0,$$

since the $\epsilon_{ijk}$ is antisymmetric and the differential operator is symmetric. This is a general result: the divergence of a curl vanishes, as does the curl of a gradient.

**3** Use the continuity equation to get density.

- $\boldsymbol{u} = (x, -y)$ Incompressible so streamfunction $\psi = xy$. Irrotational so velocity potential $\phi = \frac{1}{2}(x^2 - y^2)$. Density: $\boldsymbol{u} \cdot \nabla \rho = 0$.
- $\boldsymbol{u} = (-y, x)$ Incompressible so streamfunction $\psi = \frac{1}{2}(x^2 + y^2)$. Rotational so no velocity potential. Density: $\boldsymbol{u} \cdot \nabla \rho = 0$.
- $\boldsymbol{u} = (x, 0)$ Compressible so no streamfunction. Irrotational so velocity potential $\phi = \frac{1}{2}x^2$. Steady density satisfies $(\rho x)_x = 0$, so $\rho = A(t)/x$.

If $\rho$ is allowed to vary with time, it satisfies the equations

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0, \qquad \rho_t + (x\rho)_x = 0$$

for the first two cases and the last case respectively. The last equation can be solved using the method of characteristics.

**4** This can be done either by carrying out the surface integrals or using the divergence theorem. The latter is probably simpler:

$$N = \int_V (8xy + xz + 2yz)\,\mathrm{d}V = 8(1)(1/2)(1/2) + (1/2)(1)(1/2) + 2(1)(1/2)(1/2) = 11/4$$

since the $x$-, $y$- and $z$-integrations decouple.