Solutions IV

1 The flow is incompressible and two-dimensional so

$$u = \frac{\partial \psi}{\partial y} = \epsilon_{123}\frac{\partial \psi}{\partial x_2}, \qquad v = -\frac{\partial \psi}{\partial x} = \epsilon_{213}\frac{\partial \psi}{\partial x_1}.$$

(ii) The vorticity points out of the plane and is given by

$$\omega_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}\epsilon_{kl3}\frac{\partial \psi}{\partial x_l} = (\delta_{il}\delta_{j3} - \delta_{i3}\delta_{jl})\frac{\partial^2 \psi}{\partial x_j \partial x_l} = -\delta_{i3}\nabla^2\psi.$$

The δ_{j3} term vanishes since ψ does not depend on the x_3 coordinate (two-dimensional flow).
(iii) The velocity is

$$u_i = \epsilon_{ij3}\frac{\partial}{\partial x_j}a_{kl}x_k x_l = \epsilon_{ij3}[a_{jl}x_l + a_{kj}x_k] = \epsilon_{ij3}[a_{jl} + a_{lj}]x_l.$$

The vorticity is

$$\omega = -\frac{\partial^2}{\partial x_k \partial x_k}a_{ij}x_i x_j = -\frac{\partial}{\partial x_k}a_{ij}[\delta_{ik}x_j + x_i\delta_{jk}] = -a_{ij}[\delta_{ik}\delta_{jk} + \delta_{ik}\delta_{jk}] = -2a_{ii}.$$

The flow is irrotational when $a_{ii} = 0$.
(iv) The viscous term for incompressible flow is $\mu\nabla^2\boldsymbol{u}$. This vanishes here since the velocity field is linear.
(v) The dissipation rate for an incompressible flow is

$$\phi = 2\mu e_{ij}e_{ij}.$$

We have

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\epsilon_{ip3}(a_{pj} + a_{jp}) + \frac{1}{2}\epsilon_{jp3}(a_{pi} + a_{ip}).$$

Hence

$$\phi = 2\mu[\tfrac{1}{2}\epsilon_{ip3}\epsilon iq3(a_{pj} + a_{jp})(a_{qj} + a_{jq}) + \tfrac{1}{2}\epsilon_{ip3}\epsilon jq3(a_{pj} + a_{jp})(a_{qi} + a_{iq})].$$

The first term can be simplified but the second is hard to deal with:

$$\phi = \mu[(a_{pj} + a_{jp})(a_{pj} + a_{jp}) + \epsilon_{ip3}\epsilon jq3(a_{pj} + a_{jp})(a_{qi} + a_{iq})].$$

2 Make the following assumptions: 1) steady density field, 2) steady state, 3) inviscid fluid, and 4) uniform velocity profile and pressure. By conservation of momentum,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho u_i \,\mathrm{d}V + \int_S \rho u_i(u_j n_j)\,\mathrm{d}S = \sum F_i.$$

Consider a cylindrical fixed control volume surrounding the rocket, just covering its nozzle outlet. The component of the above equation along the direction of thrust is

$$\int_S \rho U^2 \,\mathrm{d}S = \int_S P_{atm}\,\mathrm{d}S - \int_S P\,\mathrm{d}S + F_{thrust},$$

where the P term comes from the nozzle, and the P_{atm} term comes from the opposite surface of the cylinder. All the integrands are constant. Therefore

$$F_{thrust} = \rho A U^2 + A(P - P_{atm}).$$

3 In a fluid at rest, the stress is entirely due to pressure, so that $\tau_{ij} = -p\delta_{ij}$. The momentum equation can be written as

$$\mathbf{0} = -\boldsymbol{\nabla} p + \rho \boldsymbol{\nabla}\phi.$$

Take the curl of this equation. The curl of gradients vanish, and the product rule for the last term gives $\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}\phi = \mathbf{0}$.

4 (i) In cylindrical polars, the continuity equation is

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial v_x}{\partial x} = -2\alpha + \frac{\partial v_x}{\partial x} = 0.$$

This gives $v_x = 2\alpha x + C(r) = 2\alpha x + U_0$ using the condition at $x = 0$.
(ii) At the nozzle wall, $\boldsymbol{u}\cdot\boldsymbol{n} = 0$. The equation for the nozzle wall is $F(r,x) = r - R(x) = 0$, so the normal vector is proportional to $\boldsymbol{\nabla} F = (-R', 1)$. Hence the boundary condition becomes

$$-R'(x)v_x(R(x),x) - \alpha R(x) = -(2\alpha x + U_0)R'(x) - \alpha R(x) = 0.$$

This is an ODE for the shape $R(x)$, with solution

$$R(x) = \left(\frac{R_0}{2\alpha x/U0 + 1}\right).$$

Since this cannot depend on U_0, we must have $\alpha = kU_0$.
(iii) The flow rates are

$$\begin{aligned}
\int_0^{R_0} v_x(r,0)2\pi r\,\mathrm{d}r &= \pi R_0^2 U_0,\\
\int_0^{R(L)} v_x(r,L)2\pi r\,\mathrm{d}r &= \pi R_L^2(2\alpha L + U_0) = \pi R_0^2\frac{2\alpha L + U_0}{2\alpha L/U_0 + 1} = \pi R_0^2 U_0.
\end{aligned}$$

(The x-velocity is independent of r.) The flow rates are the same since the flow is incompressible.