Midterm Solution

1 The stationary points in the (μ, x) plane are at

$$\mu^2 + x^2 = 1 \qquad \left(\frac{\mu - 2}{2}\right)^2 + x^2 = 1,$$

the circle of radius one centered at the origin and the ellipse centered at $(2,0)$ passing through the origin and $(2,1)$. The denominator is positive, and $\dot{x} > 0$ for large $|x|$. The diagram shows the stability of the points along the curves. There are four saddle-node bifurcations at $(-1,0)$ $(0,0)$, $(1,0)$ and $(4,0)$ as well as two transcritical bifurcations at $(2/3, \sqrt{5}/3)$ and $(2/3, -\sqrt{5}/3)$.

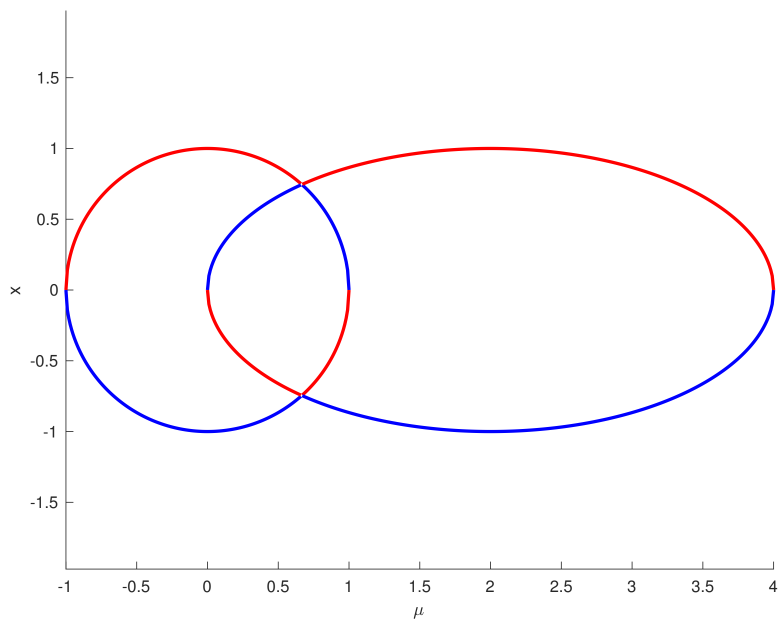

Figure 1: Bifurcation diagram in (μ, x) plane for **1**. Blue curves are stable and red curves are unstable.

2 Write $y = e^{S(x)}$ and obtain

$$S'' + S'^2 - \frac{x^2}{1 + x^2} = 0.$$

Expand the last term and assume the dominant balance is $S'^2 \sim 1$. Then $S \sim \pm x$, and $S'' \ll S'^2$ as assumed. The next term is found by writing $S = \pm x + R$, with $R \ll x$. This leads to

$$R'' \pm 2R' + R'^2 + \frac{1}{x^2} + \cdots = 0.$$

The dominant balance gives $R' \sim \mp x^{-2}/2$, so R is small as $x \to 0$. Hence the controlling behavior is $\mathrm{e}^{\pm x}$ Write $y(x) = \mathrm{e}^{\pm x} w(x)$. Then Leibniz's rule gives

$$w'' \pm 2w' + \frac{1}{1+x^2} w = 0.$$

3 Draw the graphs of $\tanh x$ and $\epsilon(x-1)^2$. One can see a root for small x and a large root. For the former, $\tanh x \sim x$ and $\epsilon(x-1)^2 \sim \epsilon$, so write $x = \epsilon x_1 + \epsilon^a x_2 + \cdots$. Then

$$\epsilon x_1 + \epsilon^a x_2 - \frac{1}{3}\epsilon^3 x_1^3 + \epsilon(1 - 2\epsilon x_1) + \cdots = 0,$$

Hence $x_1 = 1$, $a = 2$ and $x_2 = -2$. For large x, $\tanh x \sim 1 \sim \epsilon x^2$, so $1 \sim \epsilon x^2$. Try $x = \epsilon^{-1/2} + \epsilon^b X_2 + \cdots$. For large x, $\tanh x \sim 1 + \cdots$ and all the subsequent algebraic terms in the right-hand side must vanish. This means that $b = 0$ and hence

$$1 + \cdots = \epsilon[\epsilon^{-1/2} + X_2 - 1 + \cdots]^2.$$

This gives $X_2 = 1$. The two-term approximations to the roots are

$$x = \epsilon - 2\epsilon^2, \qquad x = \epsilon^{-1/2} + 1.$$

4 Skip the naive expansion. The leading-order solution is $x_0 = A(T)\mathrm{e}^{\mathrm{i}t} + A^*(T)\mathrm{e}^{-\mathrm{i}t}$ with $A(0) = 1/2$. At $O(\epsilon)$, one finds

$$x_{1tt} + x_1 + 2x_{0tT} + x_0^2 x_{0t} = 0.$$

Secular terms look like $\mathrm{e}^{\pm \mathrm{i}t}$ so look at

$$x_0^2 x_{0t} = (A\mathrm{e}^{\mathrm{i}t} + A^*\mathrm{e}^{-\mathrm{i}t})^2(\mathrm{i}A\mathrm{e}^{\mathrm{i}t} - \mathrm{i}A^*\mathrm{e}^{-\mathrm{i}t}) = \mathrm{i}A^2A^* + \cdots$$

The amplitude equation is hence

$$2A_T + |A|^2 A = 0.$$

Writing $A = R\mathrm{e}^{\mathrm{i}\Theta}$ gives $R\Theta_t = 0$ and $2R_T + R^3 = 0$. From the initial condition $\Theta(0) = 0$ and $R(0) = 1/2$. The Bernoulli equation for $R(T)$ can be solved to give

$$R(T) = \frac{1}{\sqrt{T+4}}.$$

The solution is hence

$$x(t) = \frac{2}{\sqrt{4 + \epsilon t}} \cos t + O(\epsilon)$$

uniformly for $\epsilon t = O(1)$.