MAE Examples

1 (Van Dyke; Hinch 5.12) Consider the following problem which has an outer, an inner and an inner-inner inside the inner (called a triple deck problem)

$$x^3 y' = \epsilon[(1+\epsilon)x + 2\epsilon^2]y^2$$

in $0 < x < 1$ with $y(1) = 1 - \epsilon$. Calculate two terms of the outer, then two of the inner, and finally one for the inner-inner. At each state, find the rescaling required for the next layer by examining the non-uniformity of the asymptoticness in the current layer.

Solution Outer solution: write $y = y_0 + \epsilon y_1 + \cdots$. The $O(1)$ equation is

$$x^3 y_0' = 0, \qquad y_0(1) = 1,$$

with solution $y_0 = 1$. The $O(\epsilon)$ equation is

$$x^3 y_1' = x y_0^2 = x, \qquad y_1(1) = -1,$$

with solution $y_1 = -x^{-1}$. So the two-term solution is

$$y = 1 - \frac{\epsilon}{x}.$$

This clearly breaks down when $x = O(\epsilon)$. Rescale with $x = \epsilon X$. To be careful, write $y = \epsilon^a Y$. The governing equation becomes

$$\epsilon^{2+a} X^3 Y_X = \epsilon^{2+2a}[(1+\epsilon)X + 2\epsilon]Y^2.$$

Hence $a = 0$, which is expected, because in terms of X the outer solution is $1 - X^{-1}$, which is $O(1)$. Now expand $Y = Y_0 + \epsilon Y_1 + \cdots$. The $O(1)$ equation is

$$X^3 Y_{0X} = X Y_0^2,$$

leaving the boundary condition for later. This first-order nonlinear equation can be solved by separating variables to give

$$Y_0^{-1} = X^{-1} + A.$$

For large X, we must match onto the outer solution. Informally, this means that as $X \to \infty$, $Y_0 \to 1$, so $A = 1$. Hence

$$Y_0 = \frac{X}{1+X}.$$

At $O(\epsilon)$, we have

$$X^3 Y_{1X} = (X+2)Y_0^2 + 2XY_0Y_1,$$

which can be transformed into

$$Y_{1X} - \frac{2}{X(1+X)}Y_1 = \frac{X+2}{X(1+X)^2}.$$

Using an integrating factor, we find

$$Y_1 = -\frac{X^{-2} + X^{-1} + B}{(1+X^{-1})^2}.$$

Now we use Van Dyke's rule. We find

$$\begin{aligned} y^{(0,0)} &= 1, \quad & y^{(0,1)} &= 1, \quad & y^{(1,0)} &= 1 - X^{-1}, \quad & y^{(1,1)} &= 1 - X^{-1}, \\ Y^{(0,0)} &= 1, \quad & Y^{(1,0)} &= 1, \quad & Y^{(0,1)} &= 1 - \epsilon x^{-1}, \quad & Y^{(1,1)} &= 1 - \epsilon(B + x^{-1}). \end{aligned}$$

The matching conditions give $B = 0$. Hence

$$Y = \frac{X}{1+X} - \frac{\epsilon}{1+X}.$$

This breaks down for small X when $X \sim \epsilon$. For the inner-inner solution, rescale using $X = \epsilon\xi$. However, now $y = O(X) = O(\epsilon)$, so write $y = \epsilon\eta(\xi)$. Then

$$\xi^3 \eta_\xi = [(1+\epsilon)\xi + 2]\eta^2.$$

The leading-order solution is now obtained from

$$\xi^3 \eta_{0\xi} = (\xi + 2)\eta_0^2$$

with solution

$$\eta = \frac{1}{\xi^{-2} + \xi^{-1} + C}.$$

We can match informally, because for large ξ we need to match onto X, so C must vanish. Hence

$$y = \frac{\epsilon\xi^2}{1+\xi} + \cdots$$

in the inner-inner region. The exact solution is

$$y = \frac{x^2}{\epsilon[(1+\epsilon)(x - x^2) + \epsilon^2(1 - x^2)] + (1-\epsilon)^{-1}x^2}.$$

You can check that expanding this expression for small ϵ with fixed x, X and ξ respectively gives the results above. Figure 1 shows the exact and asymptotic solutions over the interval $(0, 1)$ using semilogarithmic and logarithmic axes.

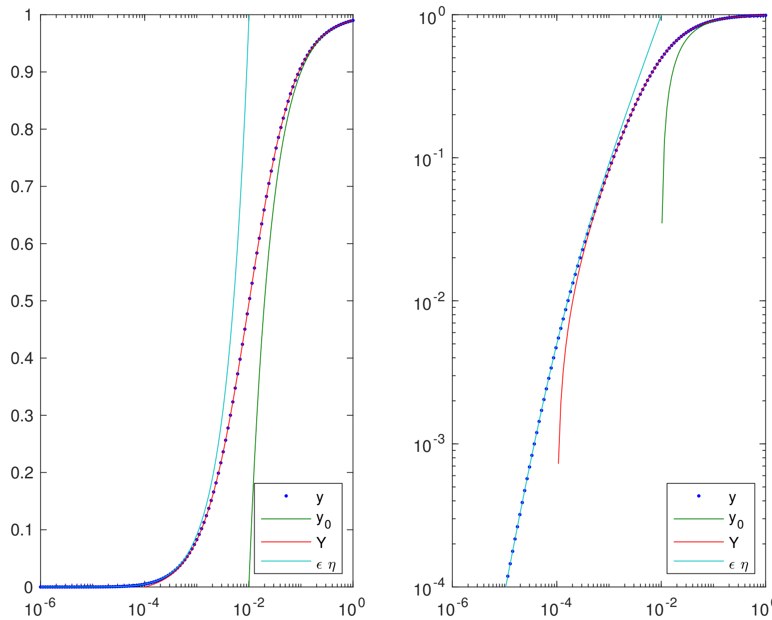

Figure 1: Exact and asymptotic solutions (two terms for the outer, two terms for the inner, one for the inner-inner) for $\epsilon = 0.01$.

2 Find a uniformly valid solution to the problem

$$\epsilon \ddot{x} + (1+t^2)\dot{x} - x(1+x) = 0, \qquad x(0) = 0, \quad x \to 1 \text{ as } t \to \infty$$

for $0 < \epsilon \ll 1$.

Solution This is a two-point BVP with a nonlinear term and an infinite domain, that can be solved using MAE. The leading-order outer equation is

$$(1+t^2)\dot{x}_0 - x_0(1+x_0) = 0.$$

Separate variables:

$$\frac{\mathrm{d}t}{1+t^2} = \frac{\mathrm{d}x_0}{x_0(1+x_0)} = \mathrm{d}x_0 \left(\frac{1}{x_0} - \frac{1}{1+x_0} \right),$$

so

$$\tan^{-1} t = \ln \frac{x_0}{1+x_0} + A.$$

Solving for x_0 and plugging in the boundary condition at infinity gives

$$x_0 = \frac{1}{2\mathrm{e}^{\pi/2 - \tan^{-1} t} - 1}.$$

We find $x_0(0) = [2\mathrm{e}^{\pi/2} - 1]^{-1}$. The inner rescaling is $t = \epsilon T$ so the leading-order inner equation is

$$\ddot{X} + \dot{X} = 0,$$

with solution $X = C(1 - \mathrm{e}^{-T})$. Matching gives $C = x_0(0)$. The uniform approximation is

$$x_u(t) = \frac{1}{2\mathrm{e}^{\pi/2 - \tan^{-1} t} - 1} - \frac{\mathrm{e}^{-t/\epsilon}}{2\mathrm{e}^{\pi/2} - 2}.$$

3 Consider the problem

$$\epsilon y'' + x^{1/2} y' - y = 0, \qquad y(0) = 0, \quad y(1) = \mathrm{e}^2.$$

Find one term of an appropriate outer expansion and one of an inner expansion, showing that they match in the correct asymptotic sense. Calculate the leading approximation to $y'(0)$.

Solution Write $y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots$. The leading-order outer solution is the solution to $x^{1/2} y_0' + y = 0$, so that $y_0 = C\mathrm{e}^{2x^{1/2}}$. There cannot be a boundary layer at $x = 1$ for the usual reason, so $y_0 = \mathrm{e}^{2x^{1/2}}$ and there is a boundary layer at the origin. The appropriate rescaling is $x = \epsilon^{2/3} X$ so that

$$y_{XX} + X^{1/2} y_X - \epsilon^{1/3} y = 0.$$

The leading-order inner solution satisfying $Y_{0XX} + X^{1/2} Y_{0X} = 0$ that vanishes at the origin is

$$Y_0 = D \int_0^X \mathrm{e}^{-2u^{3/2}/3}\, \mathrm{d}u.$$

Matching gives

$$D \int_0^\infty \mathrm{e}^{-2u^{3/2}/3}\, \mathrm{d}u = D(2/3)^{1/3}\Gamma(2/3) = 1.$$

The leading-order approximation to $y'(0)$ is

$$\epsilon^{-2/3} Y_{0X}(0) = \epsilon^{-2/3}(3/2)^{1/3}/\Gamma(2/3) \approx 0.845\epsilon^{-2/3}.$$

4 Find leading-order solutions to

$$\epsilon y'' + (\log x) y' - x(\log x) y = 0, \qquad y(\tfrac{1}{2}) = y(\tfrac{3}{2}) = 1$$

valid in different regions of the interval $(\frac{1}{2}, \frac{3}{2})$. (Hint: think carefully about the possible locations of boundary and internal layers.)

Solution The outer solution satisfies $y_0' - x y_0 = 0$, so $y_0 = A\mathrm{e}^{x^2/2}$. A boundary layer at $\frac{1}{2}$ would use the new variable $x = \frac{1}{2} + \epsilon X$, which gives the leading-order equation $Y_{XX} - (\log 2) Y_X = 0$. The solution to this equation blows up into the interior so there can be no boundary layer on the left. On the right, the new variable $x = \frac{3}{2} - \epsilon X$ gives $Y_{XX} - (\log \frac{3}{2}) Y_X = 0$. Again the solution blows up into the interior. Hence there must be an internal layer at $x = 1$. The outer solution is $\mathrm{e}^{x^2/2 - 1/8}$ for $x < 1$ and $\mathrm{e}^{x^2/2 - 9/8}$

for $x > 1$. The new variable comes from $x = 1 + \epsilon^{1/2} X$. The leading-order IL solution satisfies $Y_{XX} + XY_X = 0$, so $Y = A \operatorname{erf}(X/\sqrt{2}) + B \to \pm A + B$ as $X \to \pm\infty$. The limits of the outer solution are $\mathrm{e}^{3/8}$ to the left and $\mathrm{e}^{-5/8}$ to the right. Hence the inner solution is

$$Y = \tfrac{1}{2}(\mathrm{e}^{-5/8} + \mathrm{e}^{3/8}) + \tfrac{1}{2}(\mathrm{e}^{-5/8} - \mathrm{e}^{3/8}) \operatorname{erf}(X/\sqrt{2}).$$