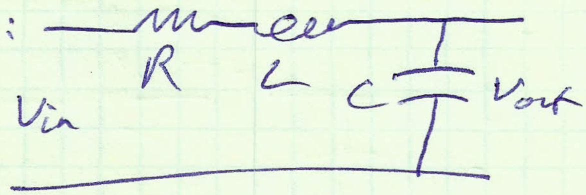
Recall the low-pass filter:



$$\frac{V_{out}}{V_{in}} = \frac{1}{1+sRC+s^2LC} = \frac{1}{1+j\omega RC-\omega^2 LC}$$

General form:

$$H(s) = \frac{1}{1+2\zeta\frac{s}{\omega_0}+\left(\frac{s}{\omega_0}\right)^2} \qquad \omega_0 = \frac{1}{\sqrt{LC}}, \quad \zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$$

note on units: $V = L\,di/dt \rightarrow [L] = V\cdot\frac{s}{A}$

$i = C\,dV/dt \rightarrow [C] = A\cdot\frac{s}{V}$

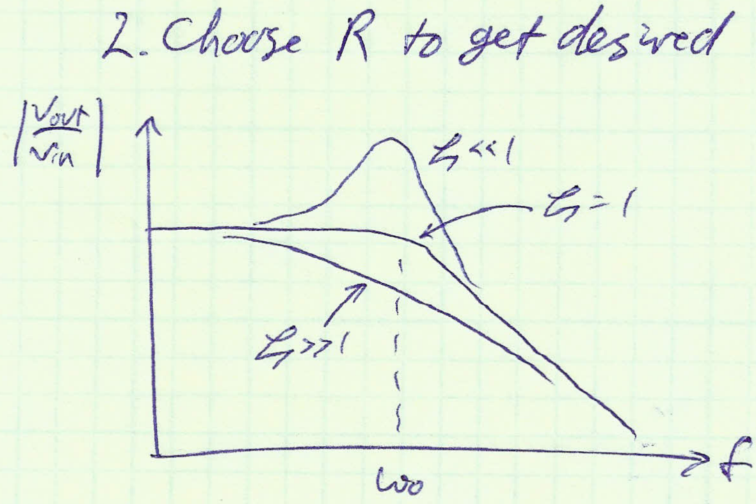
$$\left[\frac{1}{\sqrt{LC}}\right] = \frac{1}{s} \rightarrow \text{frequency}$$

$$\left[\sqrt{\frac{L}{C}}\right] = \frac{V}{A} = \text{Ohms}$$

so ζ is dimensionless

Filter design:

1. Choose L, C to get desired ω_0
2. Choose R to get desired ζ



Bode Plots (Hendrik Wade Bode - Bell Labs)

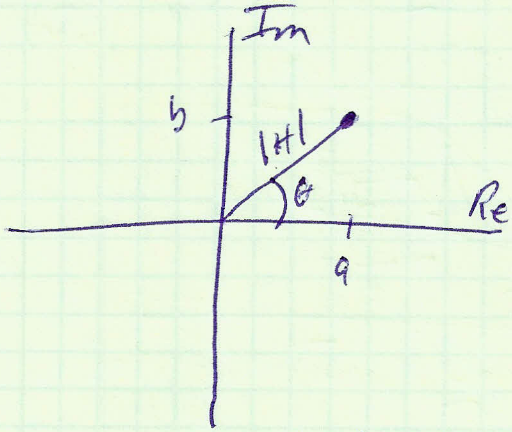
Gain $= \left|\frac{V_{out}}{V_{in}}\right|$ ... remember $|a+jb| = (a^2+b^2)^{1/2}$, $\left|\frac{1}{a+jb}\right| = \frac{1}{(a^2+b^2)^{1/2}}$

we often care about Gain^2 $\longrightarrow$ power

LPF: $|H(\omega)|^2 = \left|\frac{V_{out}}{V_{in}}\right|^2 = \frac{1}{(1-\omega^2LC)^2+(\omega RC)^2}$

however, gain is complex - it also has phase

$a+jb = |a+jb|\,e^{j\theta}$, $\theta = \tan^{-1}\left(\frac{b}{a}\right)$



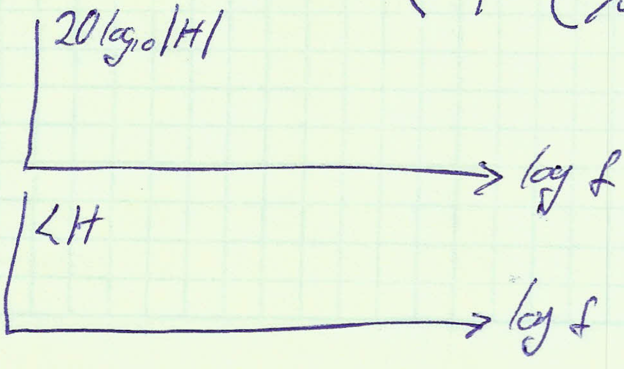
$$= \frac{1}{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2+\left(\frac{\omega 2\zeta}{\omega_0}\right)^2}$$

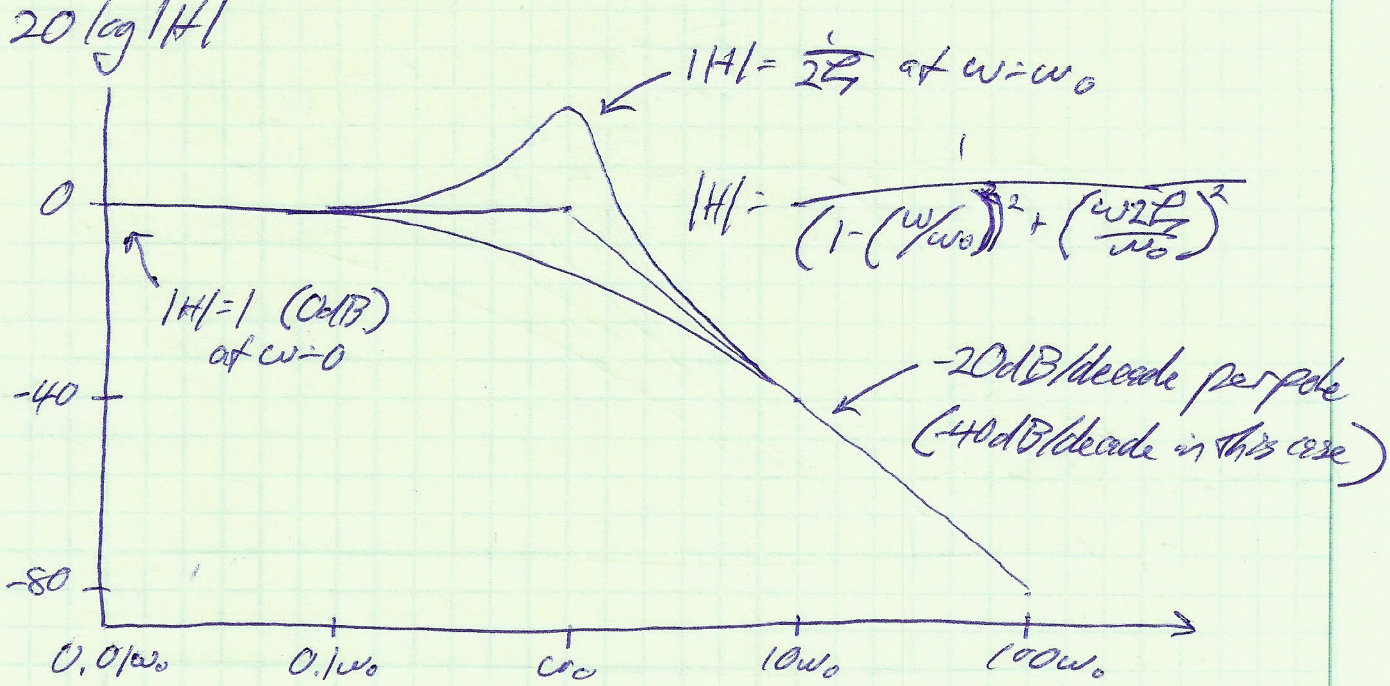
phase of $\left(\frac{1}{a+jb}\right)$ is $-\tan^{-1}\left(\frac{b}{a}\right)$

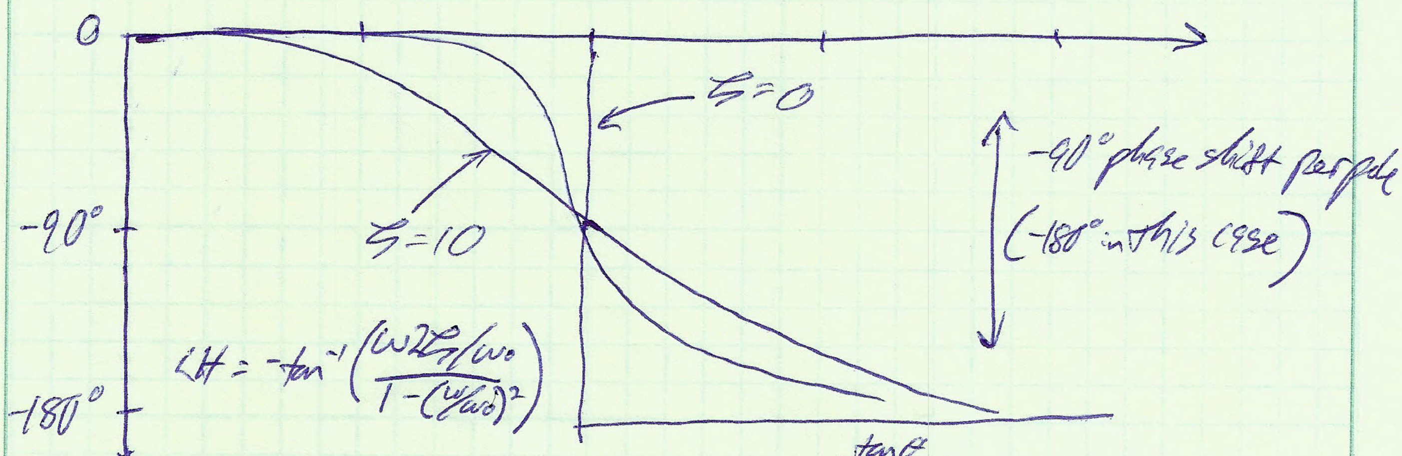
phase of $H(\omega)$ is $-\tan^{-1}\left(\frac{\omega RC}{1-\omega^2LC}\right)$

$$= -\tan^{-1}\left(\frac{\omega 2\zeta/\omega_0}{1-\left(\frac{\omega}{\omega_0}\right)^2}\right)$$

Bode Plot:

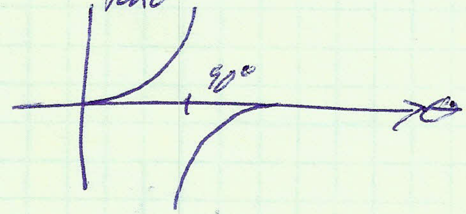






$$\angle H = -\tan^{-1}\left(\frac{\omega 2\zeta/\omega_0}{1-(\omega/\omega_0)^2}\right)$$

Note $\tan^{-1}(\infty) = 90°$



General transfer function:

$$H(s) = A\frac{(s+z_1)(s+z_2)\cdots}{(s+p_1)(s+p_2)\cdots}$$

zeros

poles

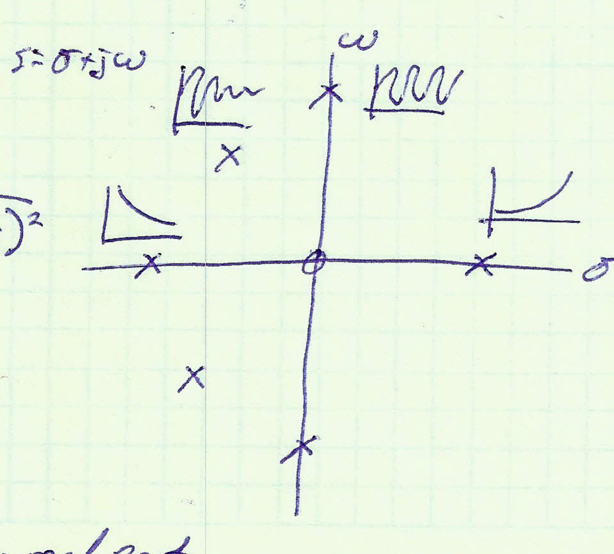
Our case: $H(s) = \frac{1}{1 + 2\zeta\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$

poles are roots of denominator

$$s = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_0$$

if $\zeta = 0$, poles at $\pm j\omega_0$

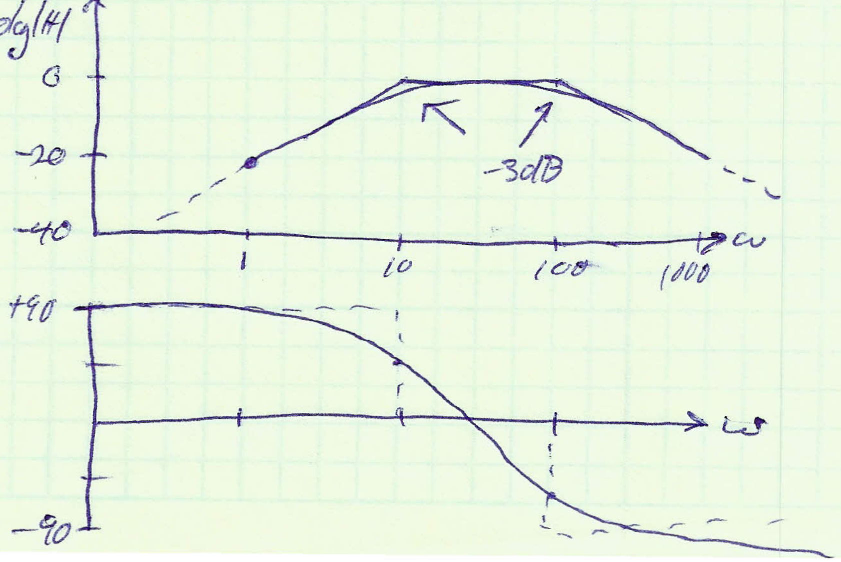
if ζ positive, poles have negative real part



How to make hand-drawn Bode plots:

1. Get $H(s)$ into form of $H(s) = A\frac{(s+z_1)(s+z_2)\cdots}{(s+p_1)(s+p_2)\cdots}$
2. Determine low frequency asymptote (value at $\omega=0$ or $\omega=1$)
3. Every pole changes slope by -20dB/decade
   Every zero changes slope by $+20$dB/decade
   (Including poles or zeros at $\omega=0$)
4. Actual curve deviates from asymptotic curve by 3dB at breakpoints (for first order) or has value of $\frac{1}{2\zeta}$ (for second order)
5. Determine low frequency asymptote of phase ($\phi = n\times 90^\circ$)
6. Phase changes by $+90^\circ$ at each zero, -90° at each pole
7. Real phase curve changes by $\pm 45^\circ$ per decade, per order
   (i.e. it is asymptotic value at $0.1\omega_0$ or $10\omega_0$)
   For Second order, can be more accurate using ζ

Example: $H(s) = \frac{100s}{(s+10)(s+100)}$ as $s\to 0$, $H(s)\to 0.1s$



Butterworth Filters (maximally flat)

Goal is to make gain as flat as possible in the pass band

$$|H(\omega)|^2 = \frac{1}{\left(1-(\omega/\omega_0)^2\right)^2 + \left(\frac{2\zeta\omega}{\omega_0}\right)^2}$$

$\frac{\partial |H(\omega)|^2}{\partial \omega}$ is difficult to solve, but $\frac{\partial \frac{1}{|H(\omega)|^2}}{\partial \omega}$ is fine

$$\frac{\partial}{\partial \omega}\left[\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(\frac{2\zeta\omega}{\omega_0}\right)^2\right] = 0 \quad \text{set } \omega_0 = 1$$

$$\frac{\partial}{\partial \omega}\left[(1-\omega^2)^2 + (2\zeta\omega)^2\right] = 0$$

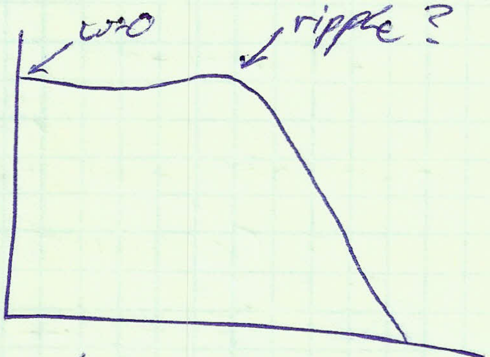
$$2(1-\omega^2)(-2\omega) + 8\zeta^2\omega = 0$$

One solution: slope is 0 at $\omega = 0$

another solution at

$$4\omega(1-\omega^2) = 8\zeta^2$$

$$\omega^2 = 1 - 2\zeta^2$$



no ripple if we set this frequency to zero

so $\zeta = \frac{1}{\sqrt{2}}$ plug this back into gain

$$|H(\omega)|^2 = \frac{1}{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(\frac{\sqrt{2}\,\omega}{\omega_0}\right)^2} \quad \text{or for } \omega_0 = 1,\ |H(\omega)|^2 = \frac{1}{(1-\omega^2)^2 + 2\omega^2}$$

$$|H(\omega)|^2 = \frac{1}{1 - 2\omega^2 + \omega^4 + 2\omega^2} = \frac{1}{1+\omega^4}$$

un-normalized: $|H(\omega)|^2 = \frac{1}{1+(\omega/\omega_0)^4}$

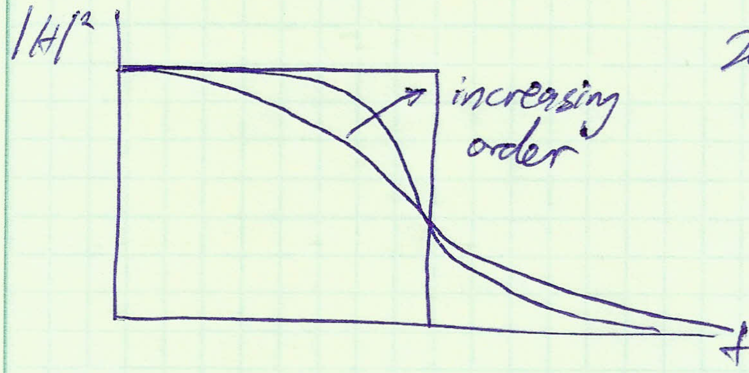
2nd order Butterworth filter

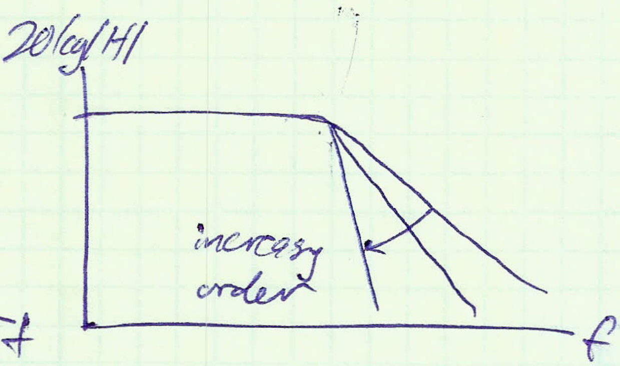
Higher order Butterworth filters:

2: $H(s) = \dfrac{1}{1+\frac{\sqrt{2}s}{\omega_0}+\frac{s^2}{\omega_0^2}}$ $\qquad |H(\omega)|^2 = \dfrac{1}{1+\left(\frac{\omega}{\omega_0}\right)^4}$

3: $H(s) = \dfrac{1}{1+\frac{2s}{\omega_0}+\frac{2s^2}{\omega_0^2}+\frac{s^3}{\omega_0^3}}$ $\qquad |H(\omega)|^2 = \dfrac{1}{1+\left(\frac{\omega}{\omega_0}\right)^6}$

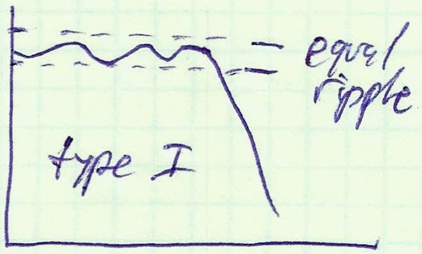
4: $\qquad |H(\omega)|^2 = \dfrac{1}{1+\left(\frac{\omega}{\omega_0}\right)^8}$ ...



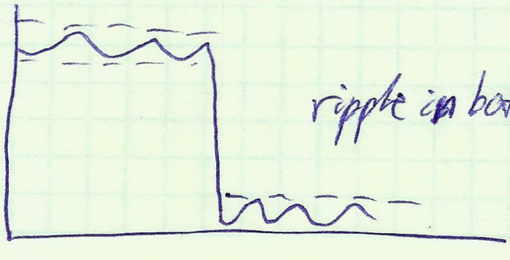


Other types of filters:

Chebyshev:



Elliptic:



ripple in both pass-band and stop-band