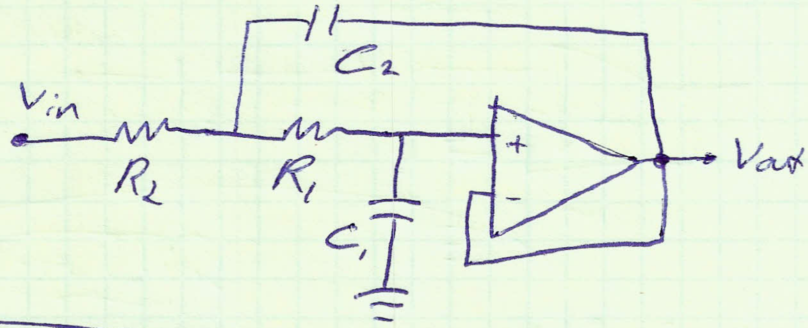
More on lab #2



last time we found:

$$\frac{V_{out}}{V_{in}} = \frac{1}{1+sC_1(R_1+R_2)+s^2C_1C_2R_1R_2}$$

How do we choose these 4 values? Start with $R_1 = R_2 = R$

$$\frac{1}{1+2sC_1R+s^2C_1C_2R^2} = \frac{1}{1+\frac{2\zeta s}{\omega_0}+\frac{s^2}{\omega_0^2}}$$

$$\omega_0^2 = \frac{1}{C_1C_2R^2}, \quad \zeta = C_1R\omega_0 = \frac{C_1R}{\sqrt{C_1C_2R^2}} = \sqrt{\frac{C_1}{C_2}}$$

set $C_1 = \zeta C$ and $C_2 = C/\zeta$ ← arbitrary choice

$$\underline{\underline{\zeta = \sqrt{\frac{C_1}{C_2}}}} = \sqrt{\frac{\zeta C}{C/\zeta}} = \zeta \quad \text{and} \quad \underline{\underline{\omega_0 = \frac{1}{RC}}}$$

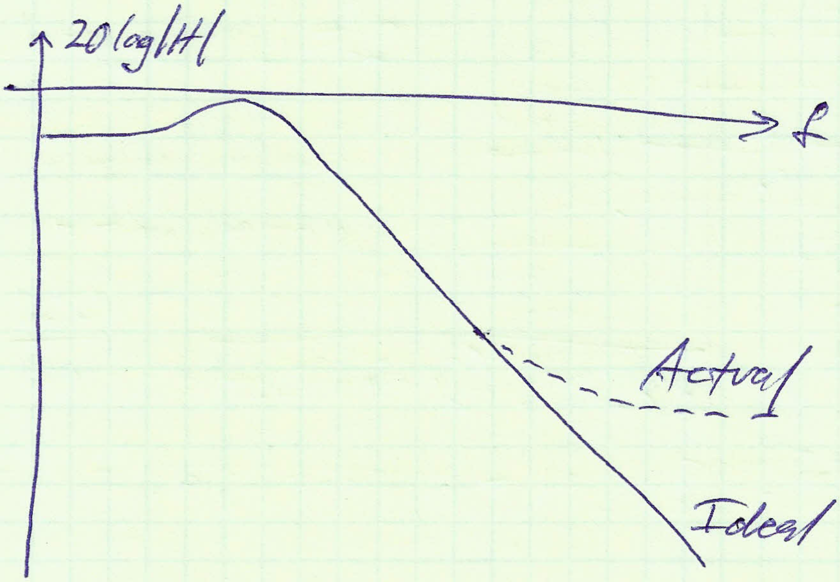
Now we only have 2 variables: R and C

What R should we choose?

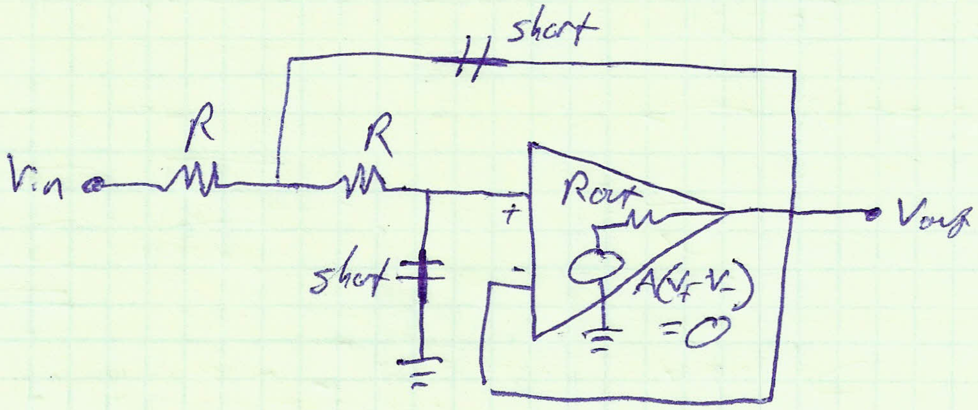
Filter is non-ideal at high frequencies, due to

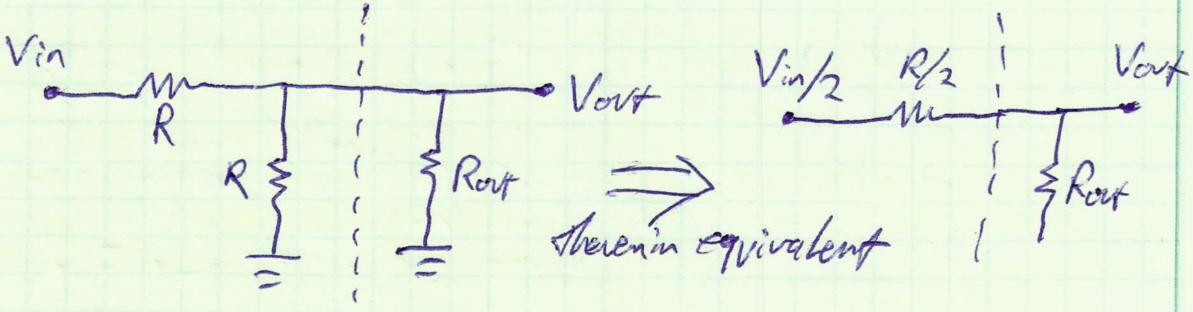
- Finite output resistance
- Reduced gain

Choose R to minimize this effect and meet -55dB gain spec at 180kHz



Assume as $\omega \to \infty$, op-amp gain $A \to 0$
and capacitors $\to$ short circuits





$$\frac{V_{out}}{V_{in}/2} = \frac{R_{out}}{R/2 + R_{out}} \simeq \frac{2R_{out}}{R}$$

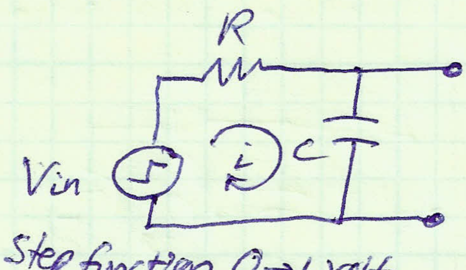
$$\frac{V_{out}}{V_{in}} = \frac{R_{out}}{R} \qquad \text{we need } \left|\frac{V_{out}}{V_{in}}\right|^2 \leq -55\,dB$$

$$\frac{R_{out}}{R} < 0.0018 \qquad \text{If } R_{out} = 50\Omega,\ R > 28\,k\Omega$$

Laplace Transform

Why do we use it? Replace complicated differential equations with simple algebra and a "look-up table"

Example:



Step function 0→1 volt

$$i(t) = C\frac{dV_{out}(t)}{dt}$$

or $V_{out}(t) = \int_0^t \frac{i(\tau)}{C}\,d\tau$

2 options:

$$1 - Ri - \int_0^t \frac{i}{C}\,d\tau = 0$$

$$1 - RC\frac{d}{dt}V_{out} - V_{out} = 0$$

neither of these are trivial!

we usually guess the solution, e.g. $V_{out} = A - Be^{-t/\tau}$ and plug it in to solve for the unknown coefficients

In this case we know the answer is $\underline{V_{out} = 1 - e^{-t/RC}}$ (starting at zero)

Easier way: $H(s) = \frac{V_{out}}{V_{in}} = \frac{1/sC}{R + 1/sC} = \frac{1/RC}{s + 1/RC}$

$$V_{out}(s) = H(s)V_{in}(s)$$

$$V_{in}(t) = U(t) \longrightarrow V_{in}(s) = \frac{1}{s}$$

$$V_{out}(s) = \frac{1/RC}{s(s + 1/RC)}$$

look this up in a table of Laplace transforms $\frac{\alpha}{s(s+\alpha)} \rightarrow (1 - e^{-\alpha t})U(t)$

$$V_{out}(t) = (1 - e^{-t/RC})U(t)$$

This is a very simple example. Imagine how much easier the Laplace Transform method is for complicated circuits!

Definition of Laplace Transform:

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

Note: there is an Inverse Laplace Transform:

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \lim_{T\to\infty} \int_{\gamma-jT}^{\gamma+jT} e^{st} F(s)\,ds$$

where $\gamma >$ all poles in $F(s)$, usually solved with Cauchy residue theorem. this is beyond the scope of this class.

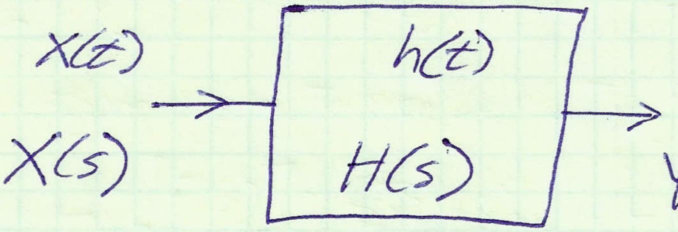
Simple example: $f(t) = e^{-t}$

$$\mathcal{L}[e^{-t}] = \int_0^\infty e^{-st} e^{-t}\,dt = \left.\frac{-e^{-(s+1)t}}{s+1}\right|_0^\infty = \frac{1}{s+1}$$

Some important Laplace transforms:

$f(t)$	$F(s)$	name
☆ → $\delta(t)$	1	unit impulse
$u(t)$	$\frac{1}{s}$	unit step
$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	exponential decay
$(1-e^{-\alpha t})u(t)$	$\frac{\alpha}{s(s+\alpha)}$	exponential approach
$\sin(\omega t)u(t)$	$\frac{\omega}{s^2+\omega^2}$	sine
$\cos(\omega t)u(t)$	$\frac{s}{s^2+\omega^2}$	cosine
$e^{-\alpha t}\sin(\omega t)u(t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$	decaying sine wave
$e^{-\alpha t}\cos(\omega t)u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$	decaying cosine wave
$f'(t)$	$sF(s)-f(0)$	differentiation
$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$	integration
☆ → $\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)\cdot G(s)$	convolution

Importance of delta function and convolution:



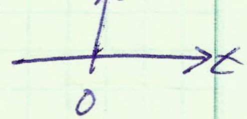
$y(t) = \int_0^t h(t-\tau)x(\tau)d\tau$

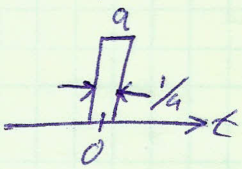
$Y(s) = H(s) \cdot X(s)$

Which would you rather solve: convolution $y(t)$ or multiplication $Y(s)$?

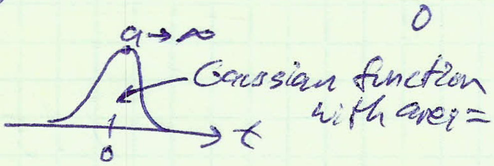
$h(t)$ is also called the impulse response function

Impulse function $\delta(t) = \begin{cases} \infty, & t=0 \\ 0, & t \neq 0 \end{cases}$ $\quad \int_{-\infty}^{\infty} \delta(t) = 1$



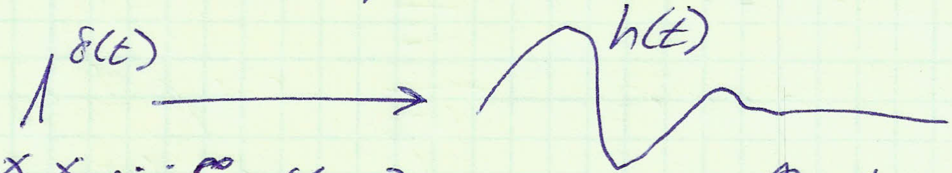


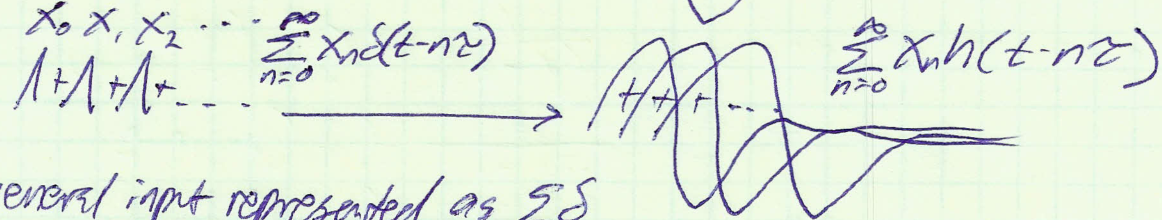
limit as $a \to \infty$ or



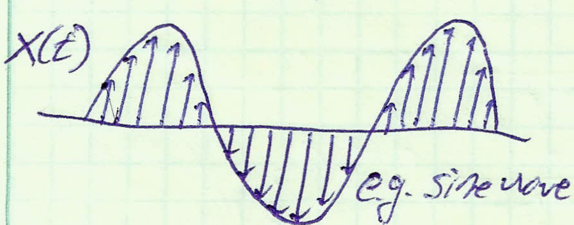
$h(t)$ is the response of the system to a "ping"

- think of a hammer tapping a mechanical system or a small voltage spike in an electrical system





general input represented as $\Sigma\delta$



output: $y(t) = \int_0^t h(t-\tau)x(\tau)d\tau$

integral form of sum of response to all previous impulses

Poles and Zeros

$$H(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{k=0}^{n} b_k s^k} = K \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{k=1}^{n} (s - p_k)}$$

$\leftarrow$ zeros z_i

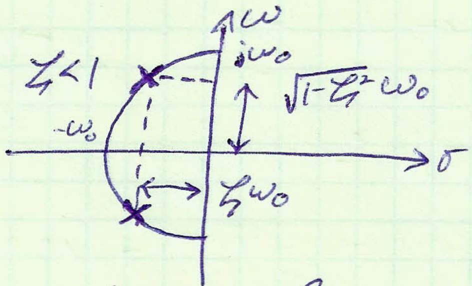
$\leftarrow$ poles p_k

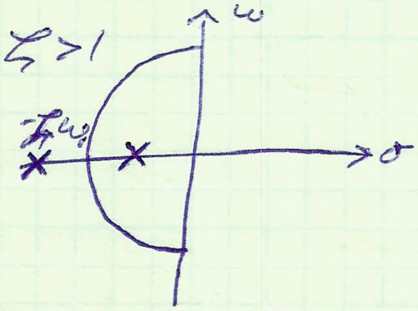
a_i, b_k are real $\rightarrow$ z_i, p_k either real, or complex conjugate pairs

e.g. $s^2 + 2\zeta s + 1 = 0$ has roots at $-\zeta \pm \sqrt{\zeta^2 - 1}$

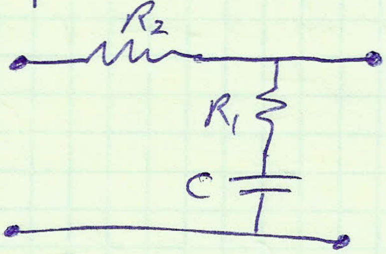
if $\zeta > 1$, the roots are both real

if $\zeta < 1$, they are $-\zeta \pm j\sqrt{1-\zeta^2}$ ie. conjugate pairs





example:



$$H(s) = \frac{R_1 + 1/sC}{R_1 + R_2 + 1/sC} = \frac{1 + sCR_1}{1 + sC(R_1 + R_2)}$$

pole at $s = -\frac{1}{C(R_1 + R_2)}$

zero at $s = -\frac{1}{CR_1}$

Constraints:

- number of zeros $\leq$ number of poles due to causality

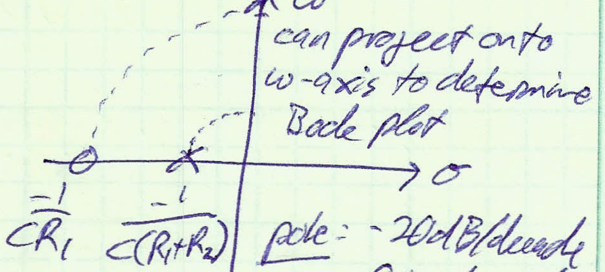
  e.g. $H(s) = s$ $\rightarrow$ derivative function would need to know future value of input

- all poles in left half plane $\rightarrow$ stable

  any poles in right half plane $\rightarrow$ unstable

  e.g. $\frac{1}{s-\alpha} \rightarrow e^{+\alpha t} u(t)$ pole at $s = +\alpha$

  exponential increase



pole: -20 dB/decade, $-90°$ phase shift

zero: $+20$ dB/decade, $+90°$ phase shift