ECE 107, Midterm 2 Solutions Spring 2010

(1) The fields in regions I and III are identical, so the energy densities are

$$W_{I} = \frac{1}{2}\varepsilon_0 |E_0^2| + \frac{1}{2\mu_0}|B_0|^2$$

$$W_{III} = W_{I}$$

The fields in region II are found from the boundary conditions. One must separate the fields into their normal and tangential components with respect to the interface.

<u>Normal components</u>

$$B_{2n} = B_{1n} = B_0 \cos\alpha$$

$$E_{2n} = \frac{\varepsilon_0}{\varepsilon_2} E_{1n} = \frac{1}{\varepsilon_r} E_0 \sin\alpha$$

<u>Tangential components</u>

$$B_{2t} = \frac{\mu_2}{\mu_0} B_{1t} = \mu_r B_0 \sin\alpha$$

$$E_{2t} = E_{1t} = E_0 \cos\alpha$$

The energy density in region II is then

$$W_{II} = \frac{1}{2}\varepsilon_2\left[E_{2n}^2 + E_{2t}^2\right] + \frac{1}{2\mu_2}\left[B_{2n}^2 + B_{2t}^2\right] \qquad \begin{aligned}\varepsilon_2 &= \varepsilon_0\varepsilon_r\\ \mu_2 &= \mu_0\mu_r\end{aligned}$$

$$= \frac{\varepsilon_0\varepsilon_r}{2}|E_0|^2\left(\cos^2\alpha + \frac{\sin^2\alpha}{\varepsilon_r^2}\right) + \frac{|B_0|^2}{2\mu_0\mu_r}\left(\mu_r^2\sin^2\alpha + \cos^2\alpha\right)$$

(2) There are a couple ways to solve this:

Method 1: Use Gauss' law to find $\vec{E}$ everywhere, then integrate to find V

$$\int_{\text{sphere}} \vec{E} \cdot d\vec{s} = \frac{Q_{enc}}{\varepsilon_0}, \qquad \vec{E} = E_R(R)\,\hat{R} \text{ by symmetry}$$

$$d\vec{s} = R^2 \sin\theta \, d\theta \, d\phi \, \hat{R}$$

since $\vec{E}$ is constant on a spherical surface,

$$\int \vec{E} \cdot d\vec{s} = E_R(R) \cdot 4\pi R^2 = \frac{Q_{enc}}{\varepsilon_0} \quad \text{where } Q_{enc} = \begin{cases} Q & \text{for } R > a \\ Q\frac{R^3}{a^3} & \text{for } R < a \end{cases}$$

$$\text{so } \vec{E} = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 R^2}\hat{R} & \text{for } R > a \\ \dfrac{QR}{4\pi\varepsilon_0 a^3}\hat{R} & \text{for } R < b \end{cases}$$

$$\text{Then } V(R=a) = -\int_{\infty}^{a} \frac{Q}{4\pi\varepsilon_0 R^2}\hat{R} \cdot \hat{R}\, dR = \frac{Q}{4\pi\varepsilon_0}\int_{\infty}^{a} \frac{dR}{R^2} = \frac{Q}{4\pi\varepsilon_0 R}$$

$$\text{and } V(0) = -\int_{\infty}^{a} \frac{Q}{4\pi\varepsilon_0 R^2}\hat{R} \cdot \hat{R}\, dR - \int_{a}^{0} \frac{QR}{4\pi\varepsilon_0 a^3}\hat{R} \cdot \hat{R}\, dR$$

$$= V(a) - \frac{Q}{4\pi\varepsilon_0 a^3} \left.\frac{R^2}{2}\right|_a^0$$

$$= \frac{Q}{4\pi\varepsilon_0 a} + \frac{Q}{4\pi\varepsilon_0 a^3}\frac{a^2}{2}$$

$$= \frac{3Q}{8\pi\varepsilon_0 a}$$

Method 2: Recognize that the potential at $R=a$ must be the same as for a point charge Q at the origin:

$$V(a) = \left.\frac{Q}{4\pi\varepsilon_0 R}\right|_{R=a} = \frac{Q}{4\pi\varepsilon_0 a}$$

for $R=0$, use

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} dV'$$

where $\vec{r} = 0$

$\vec{r}\,' = R'\hat{R}$

$\rho = \dfrac{Q}{\frac{4}{3}\pi a^3} = \text{constant}$

$dV' = R'^2 \sin\theta' dR' d\theta' d\phi'$

$$V(0) = \frac{\rho}{4\pi\varepsilon_0} \int_0^a dR \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \, \frac{R'^2}{R'}$$

$$= \frac{\rho}{4\pi\varepsilon_0} 4\pi \frac{R'^2}{2}\Bigg|_{R'=0}^{R'=a}$$

$$= \frac{3Q}{4\pi\varepsilon_0 a^3} \frac{a^2}{2}$$

$$= \frac{3Q}{8\pi\varepsilon_0 a^3}$$

3 The magnetic moment of the loop is $\vec{m} = I_2 A_2 \hat{x} = \pi a^2 I_2 \hat{x}$

The magnetic field in the solenoid is $\vec{B} = \frac{\mu N I_1}{l}\hat{z} = \mu n I_1 \hat{z}$

$$\vec{T} = \vec{m} \times \vec{B}$$

$$= (\pi a^2 I_2 \hat{x}) \times (\mu_0 n I_1 \hat{z})$$

$$= -\pi a^2 \mu_0 n I_1 I_2 \hat{y}$$

Mutual inductance

$$L = \frac{1}{I_1} \int_{\text{circle}} \vec{B} \cdot d\vec{s} = \frac{1}{I_1} \int_0^a \int_0^{2\pi} \mu n I_1 \hat{z} \cdot r dr d\phi \hat{x} = 0 \qquad \text{because } \hat{z}\cdot\hat{x} = 0$$