HW1 solutions ECE 107, Spring 2010

1. $$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}$$ $q_1 = q_2 = -1.602 \times 10^{-19}\,C$

$r = 10^{-6}\,m$

$$= 2.31 \times 10^{-16} N$$ $\varepsilon_0 = 8.85 \times 10^{-12}\,\frac{F}{m}$

$$|\vec{H}| = \frac{I}{2\pi r}$$ $I = 10^{-3}\,A$

$r = .01\,m$

$$= .016\,\frac{A}{m}$$

$$|\vec{B}| = \mu_0 |\vec{H}| = 2.0 \times 10^{-8}\,T$$

2. $$y(x,t) = A\cos\left(2\pi f t + \frac{2\pi}{\lambda}x + \phi_0\right)$$

$A = y_{max} = 40cm$

$f = 10Hz$

$\lambda = 30\,cm$

<u>Need to find ϕ_0</u>

(a) $$y(x,0) = A\cos\left(\frac{2\pi}{\lambda}x + \phi_0\right) = 0 \text{ at } x=0$$

so $A\cos\phi_0 = 0$, or $\phi_0 = \frac{\pi}{2}$

(b) $$y(3.75cm, 0) = 0$$

$$\cos\left(\frac{2\pi}{30cm} \cdot 3.75cm + \phi_0\right) = 0$$

$$2\pi \cdot (.125) + \phi_0 = \pi/2$$

$$\phi_0 = \pi/4 = .785$$

3. $V(z,t) = 5e^{-\alpha z} \sin(4\pi \times 10^9 t - 20\pi z)$

(a) $2\pi f = 4\pi \times 10^9 \text{ Hz} \Rightarrow f = 2 \text{ GHz}$

$\frac{2\pi}{\lambda} = 20\pi \text{ m}^{-1} \Rightarrow \lambda = 0.1 \text{ m}$

$V_{ph} = f \cdot \lambda = 2 \times 10^8 \text{ m/s}$

(b) $5e^{-\alpha \cdot 2} = 2V$

$-2\alpha = \ln \frac{2}{5}$

$\alpha = -\frac{1}{2} \ln \frac{2}{5} = 0.4581 \text{m}^{-1}$

4. $z_1 = 5 + 4j$

$z_2 = -3 + 3j$

(a) $|z_1| = \sqrt{5^2 + 4^2} = \sqrt{41}$ $\quad \theta_1 = \tan^{-1} \frac{4}{5} = .675$

$|z_2| = \sqrt{3^2 + 3^2} = 3\sqrt{2}$ $\quad \theta_2 = \tan^{-1}(-1) = -\frac{\pi}{4}$

so $z_1 = 6.40 e^{.675j}$

$z_2 = 4.24 e^{-.785j}$

(b) I already used eq. 1.41, so I will use eq. 1.43 only:

$|z_1| = \sqrt{z_1 z_1^*} = \sqrt{(5+4j)(5-4j)} = \sqrt{25 - 20j + 20j - 16j^2} = \sqrt{41}$

$|z_2| = \sqrt{z_2 z_2^*} = \sqrt{(-3+3j)(-3-3j)} = \sqrt{(-3)^2 + 9j - 9j - 9j^2} = 3\sqrt{2}$

4. (c) $z_1 z_2 = 6.40 \cdot 4.24\, e^{j(.675-.785)} = |z_1||z_2|e^{j(\theta_1+\theta_2)}$

$= 27.2\, e^{-.11j}$

(d) $\dfrac{z_1}{z_2} = \dfrac{|z_1|}{|z_2|} e^{j(\theta_1-\theta_2)}$

$= \dfrac{6.40}{4.24} e^{j(.675+.785)}$

$= 1.51\, e^{1.46j}$

(e) $z_1^3 = |z_1|^3 e^{j3\theta_1}$

$= (6.40)^3 e^{j3\cdot(.675)}$

$= 262.5\, e^{2.02j}$

5. Recall $v(t) = \mathrm{Re}\{\tilde{v} e^{j\omega t}\}$

$= |\tilde{v}| \cos(\omega t + \theta)$, where $\tilde{v} = |\tilde{v}| e^{j\theta}$

(a) $\tilde{v} = 9e^{-j\pi/3}$

(b) Note that $\sin(x + \pi/2) = \cos(x)$,

so $12 \sin(\omega t + \frac{\pi}{4}) = 12 \sin(\omega t - \frac{\pi}{4} + \frac{\pi}{2}) = 12\cos(\omega t - \pi/4)$

$\tilde{V} = 12e^{-j\pi/4}$

(c) $\tilde{V}(x) = 5e^{-3x} e^{-j\pi/3}$ (because $\sin(\omega t + \frac{\pi}{6}) = \cos(\omega t - \pi/3)$)

(d) $\tilde{i} = -2e^{j3\pi/4}$

5. (e) $\quad i(t) = 4\cos(\omega t - \frac{\pi}{6}) + 3\cos(\omega t - \pi/6) = 7\cos(\omega t - \pi/6)$

$$\tilde{i} = 7e^{-j\pi/6}$$

6. First, use the voltage law around the large loop:

$$0 = -V(z,t) + \frac{R'\Delta z}{2} i(z,t) + \frac{L'\Delta z}{2}\frac{\partial}{\partial t} i(z,t) + \frac{R'\Delta z}{2} i(z+\Delta z,t) + \frac{L'\Delta z}{2}\frac{\partial}{\partial t} i(z+\Delta z,t) + V(z+\Delta z,t)$$

or $$-\frac{V(z+\Delta z,t) - V(z,t)}{\Delta z} = \frac{R'}{2}\left(i(z+\Delta z,t) + i(z,t)\right) + \frac{L'}{2}\frac{\partial}{\partial t}\left(i(z+\Delta z,t) + i(z,t)\right)$$

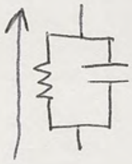
approximate $\frac{1}{2}\left(i(z+\Delta z,t) + i(z,t)\right) \approx i(z,t)$ for small Δz

take limit as $\Delta z \to 0$

$$-\frac{\partial V(z,t)}{\partial z} = R' i(z,t) + L'\frac{\partial}{\partial t} i(z,t)$$

To apply the current law we must first find the voltage V_o across the capacitor and resistor in parallel.

V_o



Applying the voltage law across the two small loops

$$V_o = V(z,t) - i(z,t)\frac{R'\Delta z}{2} - \frac{L'\Delta z}{2}\frac{\partial i(z,t)}{\partial t}$$

$$V_o = V(z+\Delta z,t) - i(z+\Delta z,t)\frac{R'\Delta z}{2} - \frac{L'\Delta z}{2}\frac{\partial i(z+\Delta z,t)}{\partial t}$$

The current law gives

$$i(z,t) - i(z+\Delta z,t) = V_o G'\Delta z + C'\Delta z\frac{\partial V_o}{\partial t}$$

$$\frac{i(z,t) - i(z+\Delta z,t)}{\Delta z} = V(z,t)G' + C'\frac{\partial V(z,t)}{\partial t} + \underbrace{\Delta z\cdot\left[-\frac{G'R'}{2}i(z,t) - \frac{G'L'}{2}\frac{\partial i(z,t)}{\partial t} - \frac{C'R'}{2}\frac{\partial i(z,t)}{\partial t} - \frac{C'L'}{2}\frac{\partial^2 i}{\partial t^2}\right]}_{\text{this term goes to zero as } \Delta z \to 0}$$

As $\Delta z \to 0$,

$$-\frac{\partial i(z,t)}{\partial z} = G'V(z,t) + C'\frac{\partial V(z,t)}{\partial t}$$

7. $R' = \frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$ $\mu_c = \mu_0$, since copper is not magnetic

$$R_s = \sqrt{\frac{\pi f \mu_c}{\sigma_c}} = \sqrt{\frac{\pi \; 10 \cdot 10^9 \, Hz \cdot 4\pi \times 10^{-7} \frac{H}{m}}{59.6 \times 10^6 \; S/m}} = .0257\,\Omega$$

$$R' = \frac{.0257\,\Omega}{2\pi}\left(\frac{1}{.91 \times 10^{-3}\,m} + \frac{1}{2.98 \times 10^{-3}\,m}\right)$$

$$= 5.89 \frac{\Omega}{m}$$

$$L' = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right) = \frac{4\pi \times 10^{-7}\frac{H}{m}}{2\pi}\ln\left(\frac{2.98}{.91}\right)$$

$$= 2.37 \times 10^{-7}\; H/m$$

$$C' = \frac{2\pi\varepsilon}{\ln(b/a)} = \frac{2\pi \cdot \left(2.1 \cdot 8.85 \times 10^{-12}\frac{F}{m}\right)}{\ln\left(\frac{2.98}{.91}\right)}$$ since $\varepsilon = 2.1\,\varepsilon_0$

$$= 9.84 \times 10^{-11}\; F/m$$

$G' = 0\,\frac{S}{m}$ because $\sigma = 0$ in the dielectric

$$Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} = 49.1 - .0097j\;\Omega$$

wavelength: $\lambda = \frac{c_0}{f\sqrt{\varepsilon_r}} = \frac{3 \times 10^8 \; m/s}{10^{10}\,Hz\,\sqrt{2.1}}$

$$= .021\,m$$

Alternatively,

$$\lambda = \frac{u_p}{f} = \frac{1}{f\sqrt{L'C'}} = .021\,m$$

phase velocity $= \frac{c_0}{\sqrt{\varepsilon_r}} = \frac{1}{\sqrt{L'C'}} = 2.07 \times 10^8 \; m/s$

8. $Z_L = R + j\omega L$, $\omega = 2\pi f = 2\pi \cdot 5\times10^6\,\text{Hz} = 3.14\times10^7\,\text{rad/s}$

(a) $$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = -\frac{(Z_0 - R) + j\omega L}{(Z_0 + R) + j\omega L}\,\frac{(Z_0 + R) - j\omega L}{(Z_0 + R) - j\omega L}$$

$$= \frac{-Z_0^2 + R^2 + \omega^2 L^2 + 2 Z_0 j\omega L}{(Z_0 + R)^2 + \omega^2 L^2}$$

$$= .552 + .313j$$

$|\Gamma| = 0.634 \leq 1$, as expected. $\theta_r = 0.516$

(b) $$S = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.634}{1 - 0.634} = 4.47$$

(c) Voltage maximum: $l_{max} = \dfrac{\theta_r \lambda}{4\pi} + \dfrac{n\lambda}{2}$ $\quad \lambda = \dfrac{3\times10^8\,\text{m/s}}{5\times10^6\,\text{Hz}} = 60\,\text{m}$

need to choose n as the integer that makes l_{max} small as possible but still positive.

In this case, $n = 0$ gives the maximum nearest the load

$$l_{max} = \frac{\theta_r \lambda}{4\pi} = 2.46\,\text{m}$$

(d) The current maximum coincides with the voltage minimum.

Since $l_{max} < \frac{\lambda}{4}$, $l_{min} = l_{max} + \frac{\lambda}{4}$

$$= 17.46\,\text{m}$$

9. $\Gamma = |\Gamma| e^{j\theta_r}$

The voltage minima are spaced $\frac{\lambda}{2}$ apart, so

$$\lambda = 2 \cdot (14\text{cm} - 4\text{cm}) = 20\text{cm}$$

and $\quad l_{max} = \dfrac{14\text{cm} + 4\text{cm}}{2} = 9\text{cm}$

is the position of the voltage maximum - it's halfway between the two minima. It must be the first maximum because $\quad l_{max} < \frac{\lambda}{2}.$

$$l_{max} = \left(\frac{\theta_r}{4\pi} + \frac{n}{2}\right)\lambda$$

$$\theta_r = \left(\frac{l_{max}}{\lambda} - \frac{n}{2}\right) \cdot 4\pi \qquad \text{Choose } n = 1 \text{ so that } -\pi < \theta_r < \pi$$

$$\theta_r = \left(\frac{9}{20} - \frac{1}{2}\right) \cdot 4\pi$$

$$= -\pi/5$$

Next, find $|\Gamma|$

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|} = 1.5$$

$$|\Gamma| = \frac{S - 1}{S + 1} = 0.2$$

So $\quad \Gamma = |\Gamma| e^{j\theta_r} = \dfrac{e^{-j\pi/5}}{5}, \qquad Z_L = Z_0 \dfrac{1 + \Gamma}{1 - \Gamma} = 67.0 - 16.4\, j\,\Omega$

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$$S = \frac{1+|\Gamma|}{1-|\Gamma|} = 3$$

$$|\Gamma| = \frac{S-1}{S+1} = \frac{1}{2}$$

but $$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{R_L - Z_0 + j\omega L}{R_L + Z_0 + j\omega L}$$

so $$|\Gamma|^2 = \frac{(R_L - Z_0)^2 + \omega^2 L^2}{(R_L + Z_0)^2 + \omega^2 L^2} = \frac{1}{4}$$

$$\omega^2 L^2 \left[1 - \tfrac{1}{4}\right] + (R_L - Z_0)^2 - \tfrac{1}{4}(R_L + Z_0)^2 = 0$$

solving for ωL,

$$\omega L = \sqrt{\tfrac{4}{3}\left(\tfrac{1}{4}(75\Omega + 50\Omega)^2 - (75\Omega - 50\Omega)^2\right)}$$

$$\omega L = 4375\ \Omega$$

$$L = \frac{66.14\Omega}{\omega}$$

$$= \frac{66.14\Omega}{2\pi \cdot 10 \cdot 10^6 \text{Hz}}$$

$$= 1.05\mu H$$

11. at $z = z_{max}$,

$$\tilde{V}(z) = V_0^+ \left(e^{-j\beta z_{max}} + |\Gamma| e^{j\beta z_{max} + j\theta_r} \right) \quad \text{where } \Gamma = |\Gamma| e^{j\theta_r}$$

$$= V_0^+ e^{-j\beta z_{max}} \left[1 + |\Gamma| e^{j(2\beta z_{max} + \theta_r)} \right]$$

$$= V_0^+ e^{-j\beta z_{max}} \left[1 + |\Gamma| \right] \quad \text{because } 2\beta z_{max} + \theta_r = -2\pi n$$

$n = 0, 1, 2 \ldots$

$$\tilde{I}(z) = \frac{V_0^+}{Z_0} \left(e^{-j\beta z_{max}} - |\Gamma| e^{j\beta z_{max} + j\theta_r} \right)$$

$$= \frac{V_0^+}{Z_0} e^{-j\beta z_{max}} \left(1 - |\Gamma| \right)$$

So the input impedance is, by definition,

$$Z_{in}(z_{max}) = \frac{\tilde{V}(z_{max})}{\tilde{I}(z_{max})} = Z_0 \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

which is purely real for lossless transmission lines, where Z_0 is real.

12. (a)

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$= \frac{(30 - 20j) - 50}{(30 - 20j) + 50}$$

$$= -\frac{1 + j}{4 - j}$$

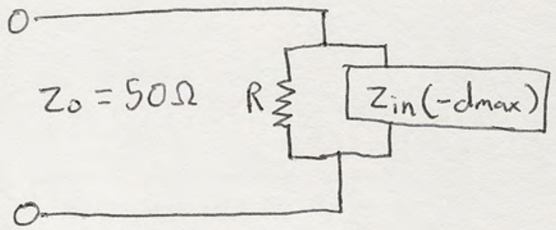
$$= -.176 - .294j$$

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

$$= \frac{1 + .343}{1 - .343}$$

$$= 2.04$$

12. (b) Using input impedance, the transmission line may be represented as follows:



where

$$Z_{in}(-d_{max}) = Z_0 \frac{Z_L + jZ_0 \tan(\beta d_{max})}{Z_0 + jZ_L \tan(\beta d_{max})}$$

To achieve a "matched load" ($\Gamma = 0$) we must have

$$Z_0 = R \,//\, Z_{in}(-d_{max}) = \frac{R\, Z_{in}(-d_{max})}{R + Z_{in}(-d_{max})}$$

or, solving for R,

$$R Z_0 + Z_0 Z_{in}(-d_{max}) = R\, Z_{in}(-d_{max})$$

$$R = \frac{Z_{in}(-d_{max})\, Z_0}{Z_{in}(-d_{max}) - Z_0}$$

Recall that at the voltage maximum,

$$\beta d_{max} = n\pi + \frac{\theta_r}{2}, \qquad \theta_r = \text{angle}(\Gamma) = -2.11$$

Because $\theta_r < 0$, $n = 1$ gives the position of the first voltage maximum.

so $\tan(\beta d_{max}) = \tan(\pi - 1.06) = -1.77$

and $Z_{in}(-d_{max}) = 102\,\Omega$

$$R = \frac{102 \cdot 50}{102 - 50} = 97.9\,\Omega$$