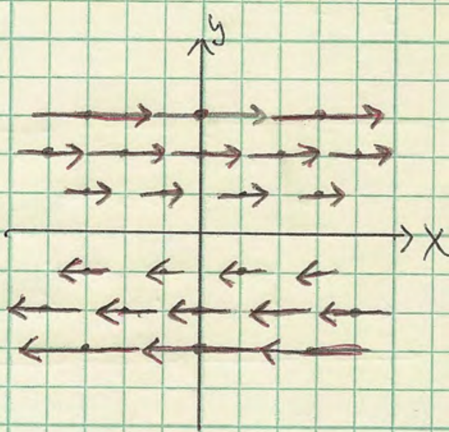
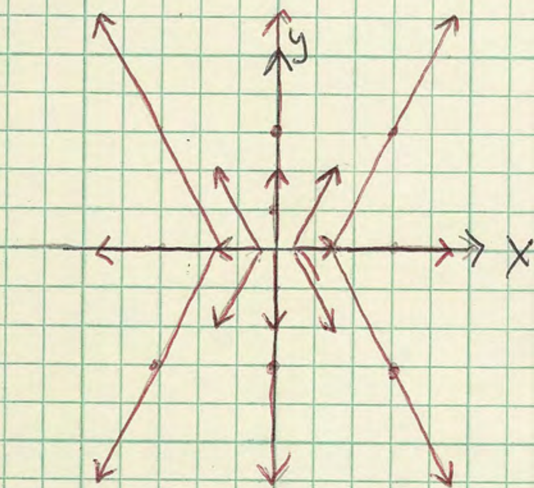
HW 3 solutions, ECE 107

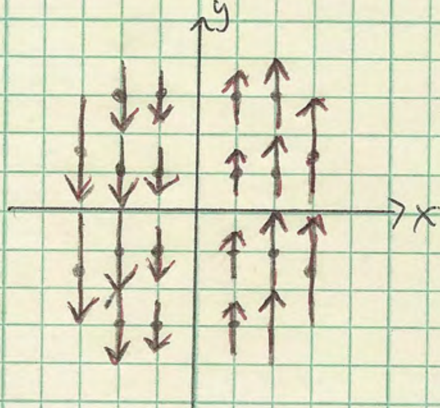
(1) (a) $\vec{E}_1 = -x\hat{y}$



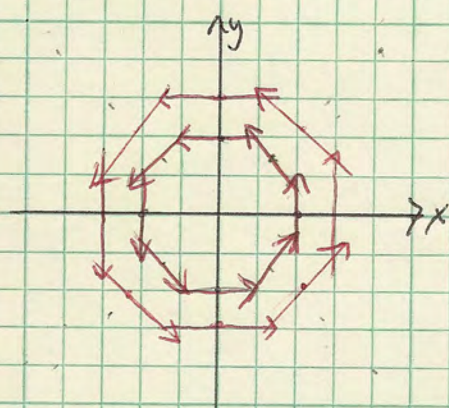
(d) $\vec{E}_4 = \hat{x}x + \hat{y}2y$



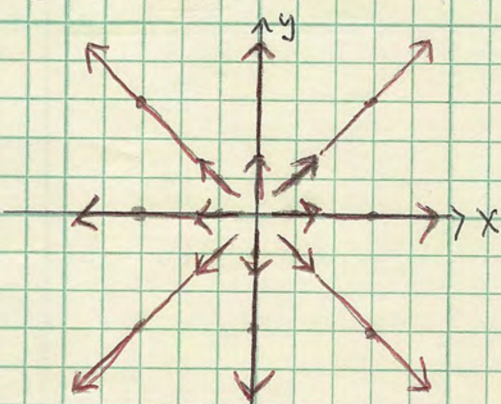
(b) $\vec{E}_2 = \hat{y}x$



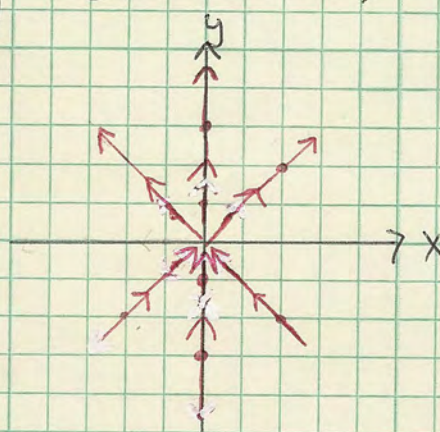
(e) $\vec{E} = \hat{\phi} r$



(c) $\vec{E}_3 = \hat{x}x + \hat{y}y$



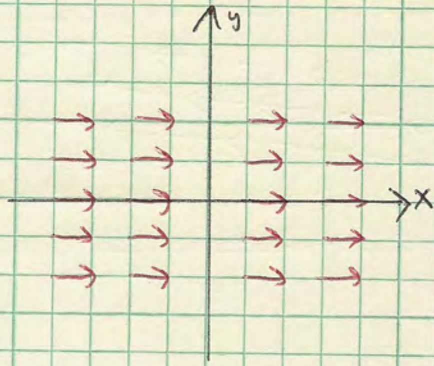
(f) $\vec{E} = \hat{r}\sin\phi$



② 3037

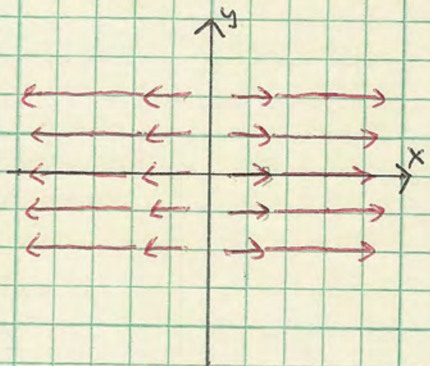
(a) $T = 10 + x$

$\nabla T = \hat{x}$



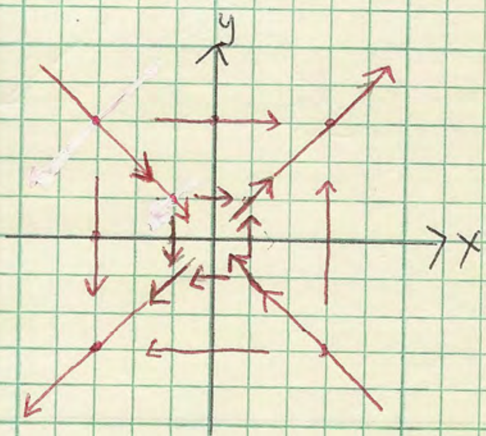
(b) $T = x^2$

$\nabla T = 2x\,\hat{x}$



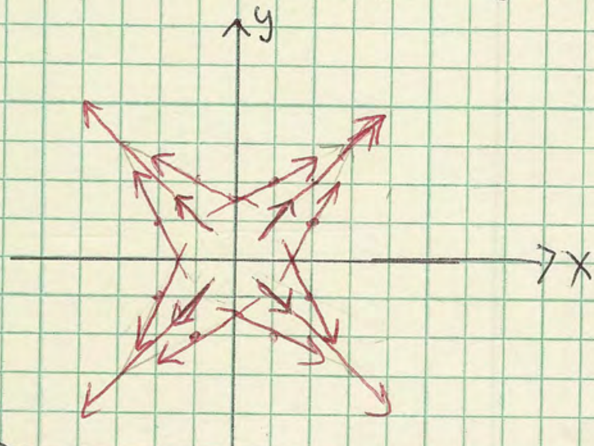
(c) $T = 100 + xy$

$\nabla T = \hat{x}y + \hat{y}x$



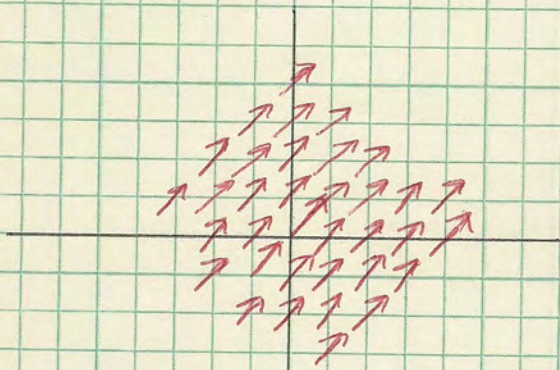
(d) $T = x^2 y^2$

$\nabla T = 2\cdot(\hat{x}\,xy^2 + \hat{y}\,yx^2)$



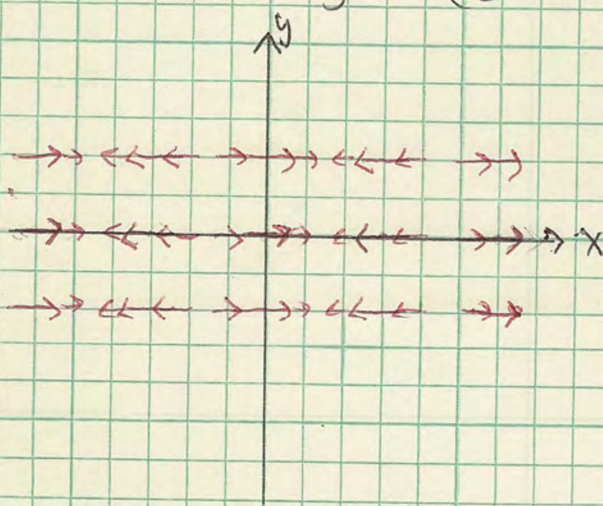
(e) $T = 20 + x + y$

$\nabla T = \hat{x} + \hat{y}$



(f) $T = 1 + \sin\left(\frac{\pi}{3}x\right)$

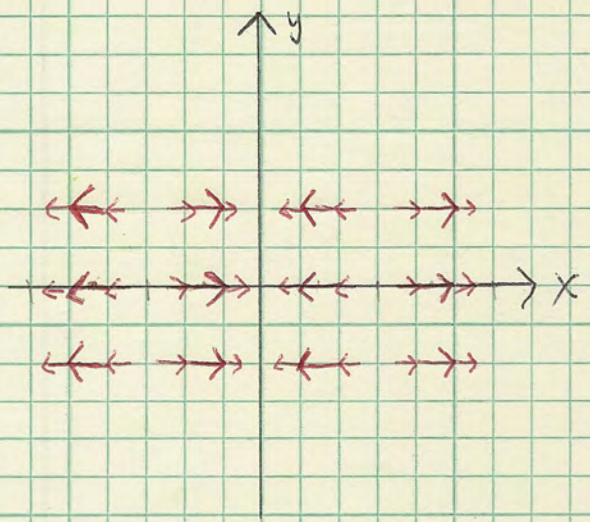
$\nabla T = \frac{\pi}{3}\cos\left(\frac{\pi}{3}x\right)\hat{x}$



② 3.37

(g) $T = 1 + \cos(\frac{\pi}{3}x)$

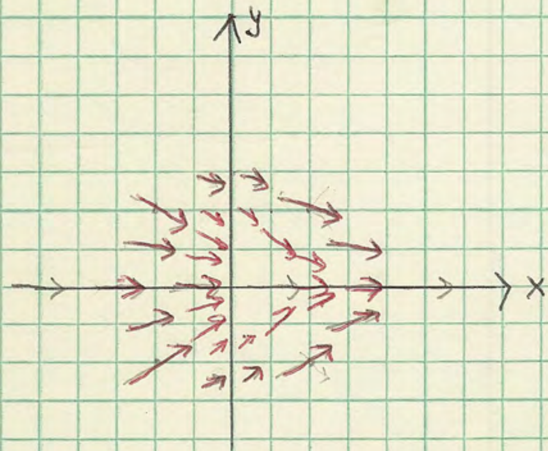
$\nabla T = -\frac{\pi}{3}\sin(\frac{\pi}{3}x)\hat{x}$



(i) $T = 15 + r\cos^2\phi$

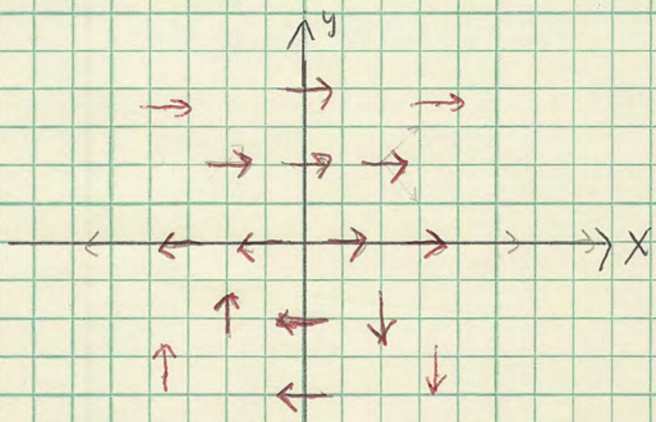
$\nabla T = \hat{r}\cos^2\phi -$

$\hat{\phi}\sin(2\phi)$



(h) $T = 15 + r\cos\phi$

$\nabla T = \hat{r}\cos\phi - \hat{\phi}\sin\phi$



3 3.44

(a) $\vec{A} = -\hat{x}\cos(x)\sin(y) + \hat{y}\sin(x)\cos(y)$

$$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}$$

$$= \sin(x)\sin(y) - \sin(x)\sin(y)$$

$$= 0$$

(b) $\vec{A} = -\hat{x}\,2\sin(2y) + \hat{y}\cos(2x)$

$$\nabla\cdot\vec{A} = -2\cos(2y)$$

(c) $\vec{A} = -\hat{x}\,xy + \hat{y}\,y^2$

$$\nabla\cdot\vec{A} = -y + 2y$$

$$= y$$

(d) $\vec{A} = -\hat{x}\cos(x) + \hat{y}\sin(y)$

$$\nabla\cdot\vec{A} = \sin(x) - \cos(y)$$

(e) $\vec{A} = \hat{x}\,x$

$$\nabla\cdot\vec{A} = 1$$

(f) $\vec{A} = \hat{x}\,xy^2$

$$\nabla\cdot\vec{A} = y^2$$

(g) $\vec{A} = \hat{x}\,xy^2 + \hat{y}\,x^2y$

$$\nabla\cdot\vec{A} = y^2 + x^2$$

$$= r^2$$

③ (h) $\vec{A} = \hat{x}\sin\left(\frac{\pi x}{10}\right) + \hat{y}\sin\left(\frac{\pi y}{10}\right)$

$$\nabla \cdot \vec{A} = -\frac{\pi}{10}\left[\cos\left(\frac{\pi x}{10}\right) + \cos\left(\frac{\pi y}{10}\right)\right]$$

(i) $\vec{A} = \hat{r}\, r + \hat{\phi}\, r\cos\phi$

$$\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r^2) + \frac{1}{r}\frac{\partial}{\partial \phi}(r\cos\phi) + 0$$

$$= 2 - \sin\phi$$

(j) $\vec{A} = \hat{r}\, r^2 + \hat{\phi}\, r^2 \sin\phi$

$$\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(r^3) + \frac{1}{r}\frac{\partial}{\partial \phi} r^2 \sin\phi$$

$$= 3r - r\cos\phi$$

$$= r(3 - \cos\phi)$$

(4) 3.46 $\vec{E} = \hat{x}\, xz - \hat{y}\, yz^2 - \hat{z}\, xy$

(a) The cube has 6 faces, so I will compute 6 integrals and add up the results to get $\oiint \vec{E} \cdot d\vec{s}$

Face 1 : positive $\hat{x}$ axis

$$d\vec{s}_1 = \hat{x}\, dy\, dz$$

$$\vec{E} \cdot d\vec{s}_1 = xz\, dy\, dz$$

$$\int \vec{E} \cdot d\vec{s}_1 = \int_{-1}^{1} dz \int_{-1}^{1} dy\, (xz) \Big|_{x=1}$$

$$= 2 \cdot \frac{z^2}{2} \Big|_{z=-1}^{z=1}$$

$$= 0$$

Face 2: negative $\hat{x}$ axis

$$d\vec{s}_2 = -\hat{x}\, dy\, dz$$

$$\int \vec{E} \cdot d\vec{s}_2 = \int_{-1}^{1} dz \int_{-1}^{1} dy\, (-xz) \Big|_{x=-1} = 0$$

Face 3: positive $\hat{y}$ axis

$$d\vec{s}_3 = \hat{y}\, dx\, dz$$

$$\int \vec{E} \cdot d\vec{s}_3 = \int_{-1}^{1} dz \int_{-1}^{1} dx\, (-yz^2) \Big|_{y=1}$$

$$= -2\, \frac{z^3}{3} \Big|_{z=-1}^{z=1}$$

$$= -\frac{4}{3}$$

Face 4 : negative $\hat{y}$ axis

$$d\vec{s}_4 = -\hat{y}\, dx\, dz$$

$$\int \vec{E} \cdot d\vec{s}_4 = \int dz \int^{1} dx\, (yz^2) \Big|_{y=-1} = -\frac{4}{3}$$

④ Face 5: positive $\hat{z}$ axis

$$d\vec{S}_5 = \hat{z}\, dx\, dy$$

$$\int \vec{E}\cdot d\vec{S}_5 = \int_{-1}^{1} dx \int_{-1}^{1} dy\, (-xy)\Big|_{z=1}$$

$$= -\frac{x^2}{2}\Big|_{x=-1}^{x=1} \cdot \frac{y^2}{2}\Big|_{y=-1}^{y=1}$$

$$= 0$$

Face 6: negative $\hat{z}$ axis

$$d\vec{S}_6 = -\hat{z}\, dx\, dy$$

$$\int \vec{E}\cdot d\vec{S}_6 = \int_{-1}^{1} dx \int_{-1}^{1} dy\, (xy)\Big|_{z=-1}$$

$$= 0$$

Total

$$\int_{cube} \vec{E}\cdot d\vec{S} = \sum_{n=1}^{6} \int \vec{E}\cdot d\vec{S}_n = -\frac{8}{3}$$

(b) $\nabla\cdot\vec{E} = z - z^2$

$$\int_{cube} \vec{\nabla}\cdot\vec{E}\, dV = \int_{-1}^{1} dx \int_{-1}^{1} dy \int_{-1}^{1} dz\, (z - z^2)$$

$$= 2\cdot 2\cdot\left(\frac{z^2}{2} - \frac{z^3}{3}\right)\Big|_{z=-1}^{z=1}$$

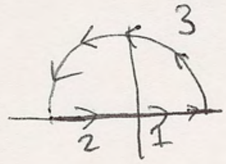
$$= 4\cdot\left(\left(\frac{1}{2} - \frac{1}{2}\right) - \left(\frac{1}{3} - \frac{-1}{3}\right)\right)$$

$$= 4\cdot\left(-\frac{2}{3}\right)$$

$$= -\frac{8}{3}$$

(5) 3.52 $\vec{B} = \hat{r}\, r\cos\phi + \hat{\phi}\sin\phi$

(a) Split the contour into 2 parts



part 1 : $d\vec{l}_1 = \hat{r}\,dr$ (because $d\vec{l}$ points away from origin)

$$\int \vec{B}\cdot d\vec{l}_1 = \int_0^2 r\cos\phi\, dr \Big|_{\phi=0} = \frac{r^2}{2}\Big|_{r=0}^{r=2} = 2$$

part 2: $d\vec{l} = -\hat{r}\,dr$ (because $d\vec{l}$ points toward the origin)

$$\int \vec{B}\cdot dl_2 = \int_0^2 -r\cos\phi\, dr \Big|_{\phi=\pi} = \frac{r^2}{2}\Big|_{r=0}^{r=2} = 2$$

part 3: $d\vec{l} = \hat{\phi}\, r\, d\phi \,\big|_{r=2} = 2\hat{\phi}\, d\phi$

$$\int \vec{B}\cdot d\vec{l}_2 = \int_0^{\pi} \sin\phi \; 2\, d\phi = 0$$

$$= -2\cos\phi \Big|_{\phi=0}^{\phi=\pi}$$

$$= 4$$

So $\oint \vec{B}\cdot d\vec{l} = 4+2+2 = 8$

(b) $\nabla\times\vec{B} = \hat{z}\,\frac{1}{r}\left[\frac{\partial}{\partial r}(r B_\phi) - \frac{\partial B_r}{\partial\phi}\right] = \frac{\hat{z}}{r}\left[\sin\phi - (-r\sin\phi)\right] = \hat{z}\sin\phi\,\frac{1+r}{r}$

$$\int \nabla\times\vec{B}\cdot d\vec{s} = \int_0^{\pi} d\phi \int_0^2 dr\, r\left(\frac{1+r}{r}\sin\phi\right)$$ (note that $d\vec{s} = r\,dr\,d\phi\,\hat{z}$)

$$= \int_0^{\pi} \sin\phi\, d\phi \int_0^2 (1+r)\, dr$$

$$= 2\cdot\left(2 + \frac{2^2}{2}\right)$$

$$= 8$$

(6) Eq. 3.106b, 3.106c

3.106b :

$$\nabla \cdot (\nabla \times \vec{A}) = \nabla \cdot \left(\hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \right)$$

$$= \left(\frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z}\right) + \left(\frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_z}{\partial y \partial x}\right) + \left(\frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y}\right)$$

$= 0$ because the terms cancel out

3.106c

$$\nabla \times (\nabla V) = \nabla \times \left(\hat{x}\frac{\partial V_x}{\partial x} + \hat{y}\frac{\partial V_y}{\partial y} + \hat{z}\frac{\partial V_z}{\partial z} \right)$$

$$= \hat{x}\left(\frac{\partial}{\partial y}\frac{\partial V_z}{\partial z} - \frac{\partial}{\partial z}\frac{\partial V_y}{\partial y}\right) + \hat{y}\left(\frac{\partial}{\partial z}\frac{\partial V_x}{\partial z} - \frac{\partial}{\partial x}\frac{\partial V_z}{\partial z}\right) + \hat{z}\left(\frac{\partial}{\partial x}\frac{\partial V_y}{\partial y} - \frac{\partial}{\partial y}\frac{\partial V_x}{\partial x}\right)$$

$= 0$ because each term cancels out individually