ECE 107, Homework ~~3~~4 solutions

(1) 4.10 From the example in the book,

$$d\vec{E} = \frac{\rho_e b}{4\pi\varepsilon_0} d\phi \frac{-\hat{r}\, b + \hat{z}\, h}{(b^2+h^2)^{3/2}} \Bigg|_{h=0} = \frac{\rho_e b}{4\pi\varepsilon_0} \frac{-\hat{r}}{b^3} d\phi$$

to find $\vec{E}$ we must integrate, but it is important to note that $\hat{r}$ is <u>not</u> a constant in the integration, as the direction changes as ϕ changes. We must use the relation $\hat{r} = \hat{x}\cos\phi + \hat{y}\sin\phi$

$$\vec{E} = -\frac{\rho_e b^2}{4\pi\varepsilon_0} \cdot \frac{1}{b^3} \int_0^{\pi} (\hat{x}\cos\phi + \hat{y}\sin\phi)\, d\phi$$

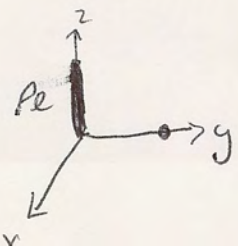
The $\hat{x}$ component vanishes, as expected from the symmetry of the problem.

$$\vec{E} = \frac{-\rho_e}{4\pi\varepsilon_0 b}\, \hat{y} \left[-\cos\phi\right]_{\phi=0}^{\phi=\pi}$$

$$= -\frac{\rho_e}{2\pi\varepsilon_0 b}\, \hat{y}$$

(2) 4.14

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho_e\, dl'}{|\vec{r}-\vec{r}\,'|^3} (\vec{r}-\vec{r}\,')$$



where $\vec{r} = 10\text{cm}\,\hat{y}$ = observation point

and $\vec{r}\,' = z'\hat{z}$ = vector pointing to charge distribution

(2) 4.14

$d\ell' = dz'$ = differential length along line charge

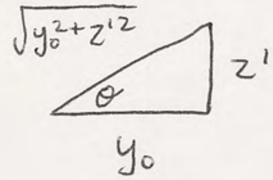
so $R = |\vec{r} - \vec{r}\,'| = \sqrt{z'^2 + (.1m)^2}$

$$\vec{E} = \frac{\rho_\ell}{4\pi\varepsilon_0}\int_{z'=0}^{z'=.05m} \frac{-z'\hat{z} + .1m\,\hat{y}}{[z'^2 + (.1m)^2]^{3/2}}\,dz' = \underbrace{\frac{\rho_\ell}{4\pi\varepsilon_0}\int \frac{-z'\hat{z}\,dz'}{(z'^2 + .1m^2)^{3/2}}}_{E_z} + \underbrace{\frac{\rho_\ell}{4\pi\varepsilon_0}\int \frac{+.1m\,\hat{y}\,dz'}{(z'^2 + .1m^2)^{3/2}}}_{E_y}$$

These two integrals require a trigonometric substitution

let $y_0 = .1m$. From the triangle drawn,

$$\tan\theta = \frac{z'}{y_0} \Rightarrow z' = y_0 \tan\theta$$



$$dz' = y_0 \sec^2\theta\, d\theta$$

$$\sin\theta = \frac{z'}{\sqrt{z'^2 + y_0^2}}, \quad \cos\theta = \frac{y_0}{\sqrt{z'^2 + y_0^2}}$$

Therefore

$$E_z = \frac{-\rho_\ell}{4\pi\varepsilon_0}\int_{z'=0}^{z'=.05m} \sin\theta \cdot \frac{\cos^2\theta}{y_0^2}\sec^2\theta\, d\theta = \frac{-\rho_\ell}{4\pi\varepsilon_0}\int \frac{\sin\theta\, d\theta}{y_0^2}$$

$$= +\frac{\rho_\ell}{4\pi\varepsilon_0 y_0^2}\cos\theta = +\frac{\rho_\ell}{4\pi\varepsilon_0 y_0^2}\frac{y_0}{\sqrt{z'^2 + y_0^2}}\Bigg|_{z'=0m}^{z'=.05m} = -7.59 \times 10^5 \frac{V}{m}$$

$$E_y = \frac{+\rho_\ell}{4\pi\varepsilon_0}\int_{z'=0}^{z'=.05m} y_0 \frac{\cos^3\theta}{y_0^3}\sec^2\theta\, d\theta = \frac{+\rho_\ell}{4\pi\varepsilon_0 y_0^2}\int \cos\theta\, d\theta$$

$$= \frac{+\rho_\ell}{4\pi\varepsilon_0 y_0^2}\sin\theta = \frac{\rho_\ell}{4\pi\varepsilon_0 y_0^2}\frac{z'}{\sqrt{z'^2 + y_0^2}}\Bigg|_{z'=0m}^{z'=.05m} = 3.22 \times 10^6 \frac{V}{m}$$

$$\vec{E}(0, 10cm, 0) = -7.59 \times 10^5 \frac{V}{m}\,\hat{z} + 3.22 \times 10^6 \frac{V}{m}\,\hat{y}$$

(3) 4.24

Choose a sphere as a surface to integrate over

$$\oint \vec{E} \cdot d\vec{s} = \frac{1}{\varepsilon_0} Q_{enclosed}$$

By symmetry, $\vec{E} = E_R \hat{R}$, where E_r depends only on R

$$d\vec{s} = R \sin\theta \, d\phi \, d\theta \, \hat{R}$$

so $$\oint \vec{E} \cdot d\vec{s} = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \, E_R R \sin\theta$$

$$= 4\pi R E_R$$

Now, for $R < a$, $Q_{enc} = 0$, so

$$4\pi R E_R = 0$$

$$\boxed{E_R = 0}$$

For $a < R < b$, $Q_{enc} = Q_1$

$$4\pi R E_R = \frac{Q_1}{\varepsilon_0}$$

$$E_R = \frac{Q_1}{4\pi\varepsilon_0 R} \Rightarrow \vec{E} = \frac{Q_1}{4\pi\varepsilon_0 R} \hat{R}$$

For $R > b$, $Q_{enc} = Q_1 + Q_2$

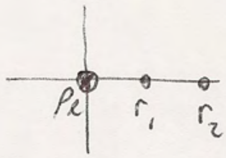
$$E_R = \frac{Q_1 + Q_2}{4\pi\varepsilon_0 R} \Rightarrow \vec{E} = \frac{Q_1 + Q_2}{4\pi\varepsilon_0 R} \hat{R}$$

(4) 4.33

The electric field from an infinite line charge is

$$\vec{E} = \frac{\rho_\ell}{2\pi\varepsilon_0 r}\hat{r}$$

Because the fields depend only on r, the two points at radial distances r_1 and r_2 may be chosen as colinear with the line charge. The potential difference is



$$\Delta V = -\int_{r_1}^{r_2} \vec{E}\cdot d\vec{\ell} \qquad d\vec{\ell} = \hat{r}\,dr$$

$$= -\frac{\rho_\ell}{2\pi\varepsilon_0}\int_{r_1}^{r_2}\frac{1}{r}\,dr$$

$$= -\frac{\rho_\ell}{2\pi\varepsilon_0}\ln\frac{r_2}{r_1}$$

(5) 4.35

At $\theta = 0°$, $R = 1\text{m}$,

$$|\vec{E}| = \frac{qd}{4\pi\varepsilon_0 (1\text{m})^3}\cdot 2 = 4\,\frac{\text{mV}}{\text{m}}$$

At $R = 2$, $\theta = 90°$

$$\vec{E} = \frac{qd}{4\pi\varepsilon_0 (2\text{m})^3}\hat{\theta}$$

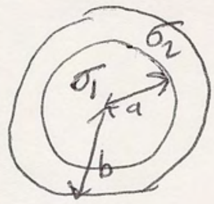
$$= \frac{4\text{mV}}{\text{m}}\cdot\frac{1}{16}\hat{\theta}$$

$$= .25\,\frac{\text{mV}}{\text{m}}\hat{\theta}$$

$$= -.25\,\frac{\text{mV}}{\text{m}}\hat{z}$$

(6) 4.44

This can be modeled as two resistors in parallel:



$$R = R_1 // R_2 = \frac{l}{\sigma_1 A_1} // \frac{l}{\sigma_2 A_2}$$

$$A_1 = \pi a^2$$

$$A_2 = \pi (b^2 - a^2)$$

$$R = \frac{\dfrac{l^2}{\sigma_1 \sigma_2 A_1 A_2}}{\dfrac{l}{\sigma_1 A_1} + \dfrac{l}{\sigma_2 A_2}} = \frac{l}{\sigma_2 A_2 + \sigma_1 A_1}$$

$$R = \frac{l}{\pi\left[\sigma_1 a^2 + (b^2 - a^2)\sigma_2\right]}$$

(7) 4.51 In free-space, $E_{t0} = E_{n0}$, so the angle is $45°$

In medium 1, $E_{t1} = E_{t0}$, but $E_{n1} = \dfrac{\varepsilon_0 E_{n0}}{3\varepsilon_0} = \frac{1}{3} E_{n0}$

So $\theta_1 = \tan^{-1}\left(\dfrac{E_{t1}}{E_{n1}}\right) = \tan^{-1}\left(\dfrac{E_{t0}}{\frac{1}{3}E_{n0}}\right) = \tan^{-1}(3) = 71.56°$

In medium 2, $E_{t2} = E_{t0}$, $E_{n2} = \dfrac{3\varepsilon_0 E_{n1}}{5\varepsilon_0}$

$$\theta_2 = \tan^{-1}\left(\frac{E_{t2}}{E_{n2}}\right) = \tan^{-1}\left(\frac{E_{t1}}{\frac{3}{5}E_{n1}}\right) = \tan^{-1}(5) = 78.69°$$

In medium 3, following the same procedure,

$$\theta_3 = \tan^{-1}(7) = 81.87°$$

(8) 4.54

a) The electric field strength is $|\vec{E}| = \frac{V}{d} = \frac{10V}{.01m} = 10^3 \frac{V}{m}$

$$F = q|\vec{E}| = 1.6\times10^{-19}C \frac{V}{d} = 1.6\times10^{-16} N$$

b) $$a = \frac{F}{m} = \frac{1.6\times10^{-16} N}{9.1\times10^{-31} kg} = 1.76\times10^{14} \frac{m}{s^2}$$

c) $$d = \frac{1}{2}at^2$$

$$t = \sqrt{2d/a}$$

$$= 1.07\times10^{-8} s$$

Alternatively, use conservation of energy

$$qV = \frac{1}{2}mv_f^2 \Rightarrow v_f = \sqrt{\frac{2qV}{m}}$$

$$t = \frac{d}{\frac{1}{2}(v_f - 0)} = \frac{d}{\frac{1}{2}\sqrt{\frac{2qV}{m}}}$$

$$= 1.07\times10^{-8} s$$

(9) 4.56

a) Boundary conditions require $E_{1t} = E_{2t}$, so

$$|\vec{E}_1| = |\vec{E}_2| = \frac{V}{d}$$

b) $$W_{e1} = \frac{1}{2}\int \varepsilon_1 |E|^2 dV = \frac{\varepsilon_1}{2} A_1 d\left(\frac{V}{d}\right)^2 = \frac{1}{2}\frac{\varepsilon_1 A_1}{d} V^2$$

$$W_{e2} = \frac{1}{2}\int \varepsilon_2 |E|^2 dV = \frac{\varepsilon_2}{2} A_2 d\left(\frac{V}{d}\right)^2 = \frac{1}{2}\frac{\varepsilon_2 A_2}{d} V^2$$

The energy stored in a capacitor is known to be $W_e = \frac{1}{2}CV^2$, so equating the expressions gives

$$C_1 = \frac{\varepsilon_1 A_1}{d}, \quad C_2 = \frac{\varepsilon_2 A_2}{d}$$

c) $$W_{e1} + W_{e2} = \frac{1}{2}\left(\frac{\varepsilon_1 A_1}{d} + \frac{\varepsilon_2 A_2}{d}\right)V^2$$

So $$C = \frac{\varepsilon_1 A_1}{d} + \frac{\varepsilon_2 A_2}{d}$$

$$= C_1 + C_2$$

(10) 4.62

Locate image charges and observe how they move when the real charge moves