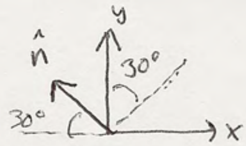
HW 5, ECE 107, Spring 2010

① 5.4



$$\vec{m} = NIA\,\hat{n} \qquad \text{where } \hat{n} = -\hat{x}\cos 30^\circ + \hat{y}\sin 30^\circ$$

$$= -\frac{\sqrt{3}}{2}\hat{x} + \frac{1}{2}\hat{y}$$

$$\vec{B} = B_0\,\hat{y}$$

$$\vec{T} = \vec{m} \times \vec{B} = NIA\left(-\frac{\sqrt{3}}{2}\hat{x} + \frac{1}{2}\hat{y}\right) \times B_0\hat{y}$$

$$= -\frac{\sqrt{3}}{2} NIAB_0\,\hat{z}$$

$$= -\frac{\sqrt{3}}{2}(20)(.5\,A)(.08\,m^2)(2.4\,T)\,\hat{z}$$

$$= -1.66\,\hat{z}\,N\cdot m$$

Clockwise rotation (by right hand rule)

② 5.14

(a) $$\vec{H}\big|_{z=0} = \underbrace{-\frac{I}{2a}\hat{z}}_{\text{bottom loop}} \underbrace{- \frac{Ia^2}{2\left(a^2+(2m)^2\right)^{3/2}}\hat{z}}_{\text{top loop}}$$

$$= -\frac{40\,A}{2\cdot 3m} - \frac{40A\,(3m)^2}{2\cdot\left(9m^2+4m^2\right)^{3/2}}\hat{z}$$

$$= -10.5\,\hat{z}\,\frac{A}{m}$$

(b) $$\vec{H}\big|_{z=1m} = -\frac{Ia^2}{2}\left(\frac{1}{(a^2+1m^2)^{3/2}} + \frac{1}{(a^2+1m^2)^{3/2}}\right)\hat{z}$$

$$= -11.4\,\frac{A}{m}\,\hat{z}$$

(c) $$\vec{H}\big|_{z=2m} = \vec{H}\big|_{z=0} = -10.5\,\frac{A}{m}\,\hat{z}$$

S.21

(3) The current in a coaxial cable flows along the outer surface of the inner conductor, and back along the inner surface of the outer conductor. This is required by the boundary conditions; it's the only current distribution that satisfies $\vec{H} = 0$ inside the conductors.

Apply Ampere's law: $\oint \vec{H} \cdot d\vec{l} = I_{enc}$

where the line integral is over a circle centered on the z-axis.

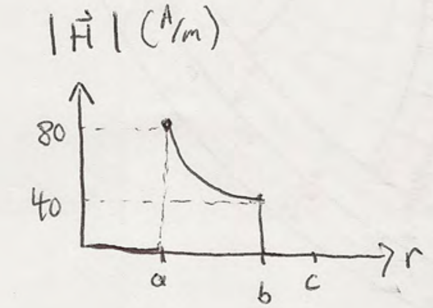
$\vec{H} = H_\phi(r)\hat{\phi}$ by symmetry, so

$$\oint \vec{H} \cdot d\vec{l} = \int_0^{2\pi} H_\phi(r)\hat{\phi} \cdot r\,d\phi\,\hat{\phi} = 2\pi r\,H_\phi(r) = I_{enc}$$

$\vec{H} = 0$ for $r \le a$ (fields are zero inside a conductor)

$\vec{H} = \frac{I}{2\pi r}\hat{\phi}$ for $a < r < b$ because $I_{enc} = I$

$\vec{H} = 0$ for $r > b$ because $I_{enc} = I - I = 0$
(no net current)



$$|\vec{H}(a)| = \frac{10A}{2\pi(.02m)} = \frac{10^3 A}{4\pi} = 80\,\frac{A}{m}$$

$$|\vec{H}(b)| = \frac{1}{2}|\vec{H}(a)| = \frac{40A}{m}$$

S.26

(4) $\nabla^2 \vec{A} = -\mu_0 \vec{J} \implies \nabla^2 A_z = -\mu_0 J_z$ because $J_x = J_y = 0$

$$A_z = \frac{\mu_0}{4\pi}\int \frac{I_z(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\,dl'$$

$dl' = dz'$

$\vec{r} = r\hat{r}$ = observation point

$\vec{r}\,' = z'\hat{z}$ = source point

$R = \vec{r} - \vec{r}\,'$

④

$$A_z = \frac{I\mu_0}{4\pi}\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \frac{dz'}{\sqrt{r^2+z'^2}} = \frac{I\mu_0}{4\pi}\left[\ln\left(2\sqrt{r^2+z'^2}+2z'\right)\right]_{-\ell/2}^{\ell/2}$$

$$= \frac{I\mu_0}{4\pi}\ln\left(\frac{\sqrt{r^2+\left(\frac{\ell}{2}\right)^2}+\frac{\ell}{2}}{\sqrt{r^2+\left(\frac{\ell}{2}\right)^2}-\frac{\ell}{2}}\right)$$

$$\vec{B} = \nabla\times\vec{A} = -\frac{\partial A_z}{\partial r}\hat{\phi}$$

$$= -\frac{I\mu_0}{4\pi}\left[\frac{2r\cdot\frac{1}{2}}{\sqrt{r^2+\left(\frac{\ell}{2}\right)^2}}\cdot\frac{1}{\sqrt{r^2+\left(\frac{\ell}{2}\right)^2}+\frac{\ell}{2}} - \frac{2r\cdot\frac{1}{2}}{\sqrt{r^2+\left(\frac{\ell}{2}\right)^2}}\frac{1}{\sqrt{r^2+\left(\frac{\ell}{2}\right)^2}-\frac{\ell}{2}}\right]\hat{\phi}$$

$$= -\frac{I\mu_0}{4\pi}\frac{r}{\sqrt{r^2+\left(\frac{\ell}{2}\right)^2}}\left[\frac{1}{\sqrt{r^2+\left(\frac{\ell}{2}\right)^2}+\frac{\ell}{2}}\cdot\underbrace{\frac{\sqrt{r^2+\left(\frac{\ell}{2}\right)^2}-\frac{\ell}{2}}{\sqrt{r^2+\left(\frac{\ell}{2}\right)^2}-\frac{\ell}{2}}}_{=1} - \frac{1}{\sqrt{r^2+\left(\frac{\ell}{2}\right)^2}-\frac{\ell}{2}}\cdot\underbrace{\frac{\sqrt{r^2+\left(\frac{\ell}{2}\right)^2}+\frac{\ell}{2}}{\sqrt{r^2+\left(\frac{\ell}{2}\right)^2}+\frac{\ell}{2}}}_{=1}\right]\hat{\phi}$$

$$= -\frac{I\mu_0}{4\pi}\frac{r}{\sqrt{r^2+\left(\frac{\ell}{2}\right)^2}}\left[\frac{-\frac{\ell}{2}-\frac{\ell}{2}}{\left(r^2+\left(\frac{\ell}{2}\right)^2\right)-\left(\frac{\ell}{2}\right)^2}\right]\hat{\phi}$$

$$= \frac{\mu_0 I \ell}{4\pi\sqrt{r^2+(\ell/2)^2}}\hat{\phi}$$

$$= \frac{\mu_0 I \ell}{2\pi\sqrt{4r^2+\ell^2}}\hat{\phi}$$

5.31

(5) $B = \mu_0 H$

$$1.5\,T = \left(4\pi \times 10^{-7} \tfrac{H}{m}\right) \cdot \left(8.5 \times 10^{28} \tfrac{\text{atoms}}{m^3} \cdot N_e \tfrac{\text{electrons}}{\text{atom}} \cdot 9.27 \times 10^{-24} A\cdot m^2\right)$$

Solve for N_e:

$$N_e = 1.51 \text{ electrons}/\text{atom}$$

5.36

(6) Apply boundary conditions $B_{1n} = B_{2n}$

$$|\vec{B}_1| \cos\theta_1 = |\vec{B}_2| \cos\theta_2$$

$$|\vec{B}_2| \cos\theta_3 = |\vec{B}_2| \cos\theta_2 = |\vec{B}_3| \cos\theta_4$$ as $\theta_2 = \theta_3$ (alternate interior angles)

So $|\vec{B}_1| \cos\theta_1 = |\vec{B}_3| \cos\theta_4$

Now apply $H_{1t} = H_{2t}$

$$\frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2}$$

so $$\frac{|\vec{B}_1| \sin\theta_1}{\mu_1} = \frac{|\vec{B}_2| \sin\theta_2}{\mu_2} = \frac{|\vec{B}_3| \sin\theta_4}{\mu_3}$$

combining these equations gives
(dividing)

$$\frac{|\vec{B}_1| \sin\theta_1 \cdot \frac{1}{\mu_1}}{|\vec{B}_1| \cos\theta_1} = \frac{|\vec{B}_3| \sin\theta_3 \, \frac{1}{\mu_3}}{|\vec{B}_3| \cos\theta_3} \Rightarrow \frac{\tan\theta_1}{\mu_1} = \frac{\tan\theta_3}{\mu_3}$$

independent of μ_2 !!

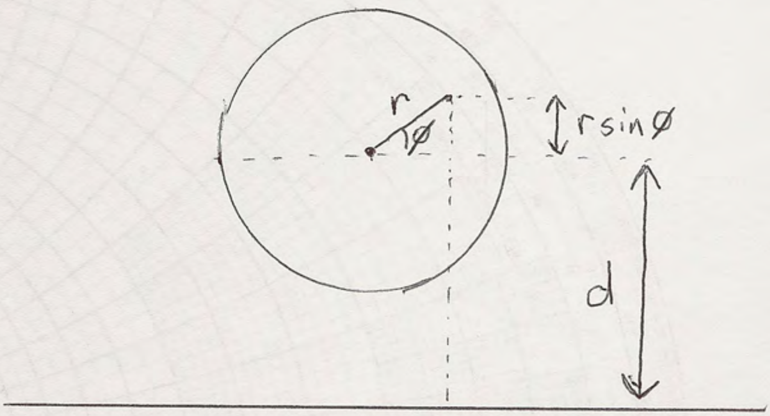
5.41

7

$$L = \frac{1}{I} \int \vec{B} \cdot d\vec{s}$$

integrate over the circle; $\vec{B}$ is the field from the wire

At any point (r, ϕ) in the circle, the distance to the wire is $d + r\sin\phi$.



The magnetic field, in this coordinate system, is given as

$$\vec{B} = \frac{\mu_0 I}{2\pi (d + r\sin\phi)} \hat{z}$$

and $d\vec{s} = r\,dr\,d\phi\,\hat{z}$ for the circle.

$$L = \frac{1}{I} \cdot \int_0^{2\pi} d\phi \int_0^{a} \frac{\mu_0 I}{2\pi (d + r\sin\phi)} r\,dr$$

$$= \frac{\mu_0}{2\pi} \int_0^{2\pi} d\phi \int_0^{a} \frac{r\,dr}{d + r\sin\phi}$$

This difficult integral is worked out below

Do the phi integration first by the substitution $z = e^{j\phi}$

so $\sin\phi = \frac{1}{2j}\left(z + \frac{1}{z}\right)$ and $dz = je^{j\phi}d\phi = jz\,d\phi$

The integral becomes

$$\frac{\mu_0}{2\pi}\int_0^a r\,dr \int_0^{2\pi} \frac{d\phi}{d + r\sin\phi} = \frac{\mu_0}{2\pi}\int_0^a r\,dr \oint \frac{1}{d + \frac{r\left(z+\frac{1}{z}\right)}{2j}}\frac{dz}{jz} = \frac{\mu_0}{2\pi}\int_0^a r\,dr \oint \frac{2\,dz}{rz^2 + 2jdz + r}$$

The integration contour is simply the unit circle in the complex z-plane. (Remember, $z = e^{j\phi}$ changes in phase but not magnitude as ϕ goes from 0 to 2π)

The integrand has poles at $z_{pole} = \frac{-2jd \pm \sqrt{-4d^2 - 4r^2}}{2r} = \frac{-jd \pm j\sqrt{d^2+r^2}}{r}$

One of these poles lies inside the unit circle. We may then use the residue theorem from complex analysis to equate the line integral to $2\pi j \cdot \left(\text{residue of integrand at } z_{pole} = \frac{j}{r}\cdot\left(-d + \sqrt{d^2+r^2}\right)\right)$

Hence,

$$L = \frac{\mu_0}{2\pi}\int_0^a r\,dr \cdot 2\pi j \left.\frac{2}{\frac{d}{dz}\left(rz^2 + 2jzd + r\right)}\right|_{z = \frac{j}{r}\left(-d+\sqrt{r^2+d^2}\right)}$$

$$= \frac{\mu_0}{2\pi}\int_0^a r\,dr\; 2\pi j \left.\frac{2}{2rz + 2jd}\right|_{z = \frac{j}{r}\left(-d+\sqrt{r^2+d^2}\right)} = \frac{\mu_0}{2\pi}\int_0^a r\,dr \frac{2\pi}{\sqrt{r^2+d^2}}$$

Now solve the integral over r:

$$L = \mu_0 \int_0^a \frac{r\,dr}{\sqrt{r^2+d^2}} \qquad \text{let } u = r^2 + d^2, \quad du = 2r\,dr$$

$$= \mu_0 \int \frac{1}{\sqrt{u}}\frac{du}{2} = \frac{\mu_0}{2}\frac{u^{1/2}}{1/2} = \mu_0 \left.\sqrt{r^2+d^2}\right|_0^a$$

So $L = \mu_0\sqrt{a^2+d^2}$