Solution of First-Order Linear Differential Equation

The solution to a first-order linear differential equation with constant coefficients,

$$a_1 \frac{dX}{dt} + a_0 X = f(t) \ ,$$

is $X = X_n + X_f$, where X_n and X_f are, respectively, natural and forced responses of the system.

The natural response, X_n, is the solution to the homogeneous equation (RHS=0):

$$a_1 \frac{dX}{dt} + a_0 X = 0$$

The functional form of X_n is $X_n = Ke^{st}$ (K and s are constants). Value of s can be found by substituting the functional form in the homogeneous differential equation:

$$a_1 \frac{dKe^{st}}{dt} + a_0 K e^{st} = 0$$

$$a_1 K s e^{st} + a_0 K e^{st} = 0 \quad \rightarrow \quad a_1 s + a_0 = 0 \quad \rightarrow \quad s = -\frac{a_0}{a_1}$$

Constant K is found from initial conditions. As the initial condition applies to X not X_n, K should be found after X_f is calculated.

Some functional forms of the forced solution, X_f, are given in the table below. To find X_f, the functional form is substituted in the original differential equation and the constant coefficients of the functional form are found.

Trial Functions for Forced Response

$f(t)^\star$	Trial Function†
a	A
$at + b$	$At + B$
$at^n + bt^{n-1} + ...$	$At^n + Bt^{n-1} + ...$
$ae^{\sigma t}$	$Ae^{\sigma t}$
$a\cos(\omega t) + b\sin(\omega t)$	$A\cos(\omega t) + B\sin(\omega t)$

* Constants a and b are used to define the general form of $f(t)$.
† Constants A and B in trial functions are found from substitution in the differential equation.

Example: Solve the differential equation below with the initial condition of v(t=0)=1.

$$\frac{dv}{dt} + 5v = 10$$

The solution is $v = v_n + v_f$. v_n is the solution to the homogeneous equation:

$$\frac{dv}{dt} + 5v = 0$$

Using the trial function $v_n = Ke^{st}$, we find:

$$\frac{d}{dt}\left(Ke^{st}\right) + 5Ke^{st} = 0$$

$$Kse^{st} + 5Ke^{st} = 0 \qquad \rightarrow \qquad s = -5$$

$$v_n = Ke^{-5t}$$

K will be found later from the initial condition.

To find v_f, we note that $f(t) = 10 = \mathit{constant}$. From the table of the trial functions, we find that the functional form of $v_f = A = \mathit{constant}$. Substituting for v_f in the differential equation, we get:

$$\frac{dv}{dt} + 5v = 10 \qquad \rightarrow \qquad \frac{dA}{dt} + 5A = 10$$

$$0 + 5A = 10 \qquad \rightarrow \qquad v_f = A = 2$$

$$v = v_n + v_f = Ke^{-5t} + 2$$

We now use the initial condition, $v(t = 0) = 1$, to find K:

$$v(t = 0) = \left[Ke^{-5t} + 2\right]_{t=0} = 1 \qquad \rightarrow \qquad K + 2 = 1 \qquad \rightarrow \qquad K = -1$$

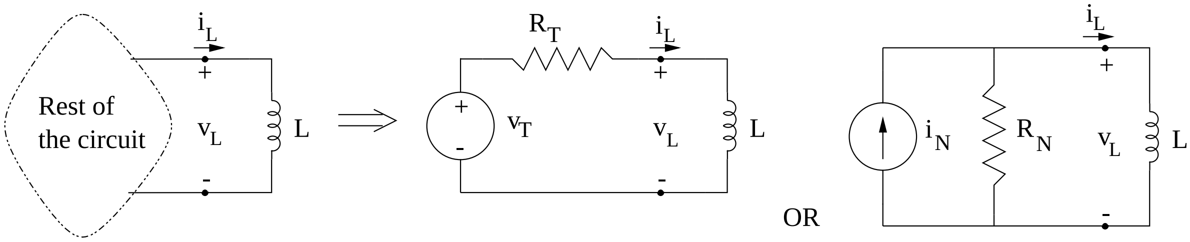
$$v = -e^{-5t} + 2$$

First-Order Circuits

First order circuits include only one capacitor or inductor (after using series/parallel reduction). The solution of these circuits results in a "first-order" linear differential equation with constant coefficients. It requires <u>one</u> initial condition.

First-order RL Circuits

Consider a first-order circuit containing only one inductor. The rest of the circuit contains only resistors and voltage and current sources. Therefore, the rest of the circuit is a two-terminal resistive subcircuit and can be reduced to a Thevenin or Norton form as shown. Note that it is possible that the rest of circuit reduced to only a resistor, *i.e.*, $v_T = 0$.



Solving the Thevenin equivalent circuit, we get:

$$\text{KVL:} \qquad R_T i_L + v_L - v_T = 0$$

$$i\text{-}v \text{ Eq.:} \qquad v_L = L\,\frac{di_L}{dt}$$

Substituting for v_L in KVL, we arrive at the general form of differential equation that describes first-order RL circuits:

$$L\,\frac{di_L}{dt} + R_T i_L = v_T$$

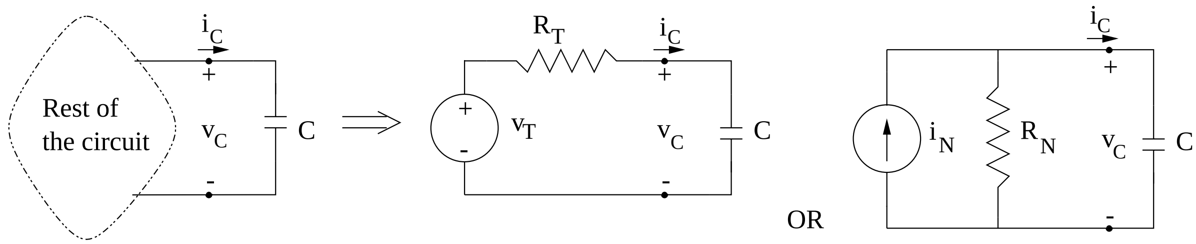
If we use the Norton form for the resistive subcircuit, we will arrive at

$$\frac{L}{R_N}\,\frac{di_L}{dt} + i_L = i_N$$

which is exactly identical to the Thevenin form ($R_N = R_T$ and $v_T = R_T i_N$).

First-order RC Circuits

Consider a first-order circuit containing only one capacitor. Similar to a first-order RL circuit, we can reduce the rest of the circuit to a Thevenin or Norton form as shown.



Solving the Thevenin equivalent circuit, we get:

$$\text{KVL:} \qquad R_T i_C + v_C - v_T = 0$$

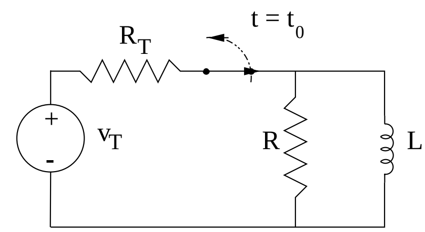
$$i\text{-}v \text{ Eq.:} \qquad i_C = C\,\frac{dv_C}{dt}$$

Substituting for i_C in KVL, we arrive at the general form of differential equation that describes first-order RC circuits:

$$R_T C\,\frac{dv_C}{dt} + v_C = v_T$$

The Norton form will also lead to an equation similar to above.

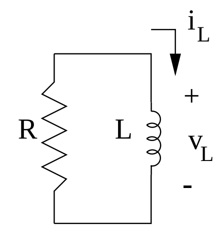
Natural Response of First-Order Circuits



As the natural response of a circuit is generic to the circuit and is independent of the driving sources, we consider the natural response (no sources) first.

In these circuits, the inductor or the capacitor is "charged" with a voltage or current source, a switch opens or closes removing the source from the circuit, and letting the capacitor or inductor discharge in a resistor.

Natural Response of RL Circuits



A generic RL circuit with an initial condition of $i_L(t = t_0^+) = i_0$ is shown. For $t > t_0$, we have:

$$\begin{aligned}
\text{KVL:} \quad & v_L + Ri_L = 0 \\
& L\frac{di_L}{dt} + Ri_L = 0 \\
\text{I.C.:} \quad & i_L(t = t_0^+) = i_0
\end{aligned}$$

It is a good idea to write down the initial conditions (again) next to the differential equation.

Since the RHS of the equation is zero, there is no forced solution. The functional form of the natural solution is $i_L = Ke^{st}$. Substituting this functional form in the differential equation (to find s), we get:

$$i_L = Ke^{st} \qquad \frac{di_L}{dt} = Kse^{st}$$

$$LKse^{st} + RKe^{st} = 0 \quad \rightarrow \quad Ls + R = 0 \quad \rightarrow \quad s = -\frac{R}{L}$$

$$i_L = Ke^{-(R/L)t}$$

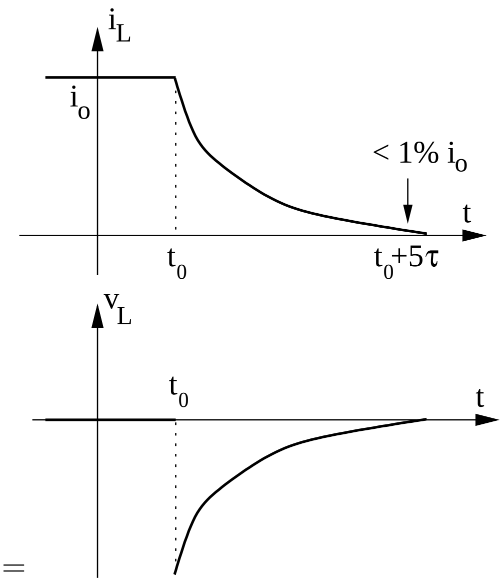
Define **time constant**, τ, as $\quad \tau = \dfrac{-1}{s} = \dfrac{L}{R} \quad \rightarrow \quad i_L = Ke^{-\frac{t}{\tau}}$

Constant K is found from the initial condition, $i_L(t = t_0^+) = i_0$:

$$i_L(t = t_0^+) = i_0 = Ke^{-\frac{t_0}{\tau}} \quad \rightarrow \quad K = i_0 e^{+\frac{t_0}{\tau}}$$

Thus:

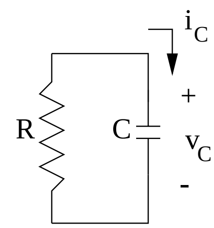
$$i_L = i_0 e^{-\dfrac{t-t_0}{\tau}} \quad \text{and} \quad v_L = -Ri_0 e^{-\dfrac{t-t_0}{\tau}}$$



Current and voltage waveforms are plotted in the figure. After several time constants (5τ is adequate), the current and voltage decay away and the circuit reaches its steady condition.

Natural Response of RC Circuits

A generic RC circuit with an initial condition of $v_C(t = t_0^+) = v_0$ is shown. For $t > t_0$, we have:



KVL: $\quad Ri_c + v_C = 0$

$$RC\frac{dv_C}{dt} + v_C = 0$$

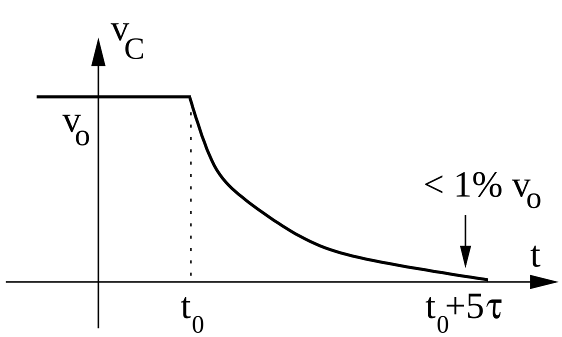
I.C.: $\quad v_C(t = t_0^+) = v_0$

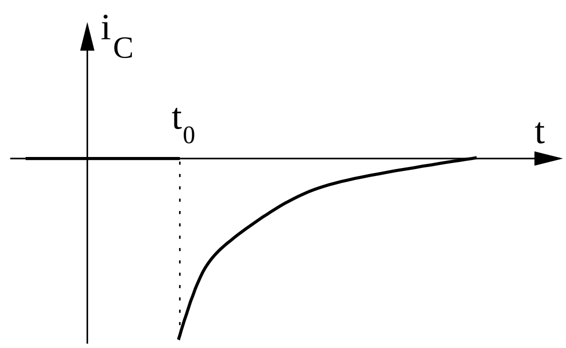
Substituting this functional form for v_C in the differential equation we find $s = -1/(RC)$.

Define **time constant**, τ, as $\quad \tau = \dfrac{-1}{s} = RC \quad \rightarrow \quad v_C = Ke^{-\dfrac{t}{\tau}}$

Constant K is found from the initial condition, $v_C(t = t_0^+) = v_0$ and we arrive at:

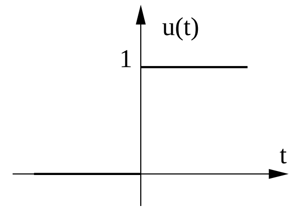
$$v_C = v_0 e^{-\dfrac{t-t_0}{\tau}} \quad \text{and} \quad i_C = -\frac{v_0}{R} e^{-\dfrac{t-t_0}{\tau}}$$





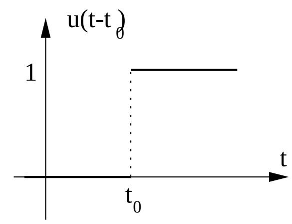
First-Order Circuits with DC sources (Step Response)

Unit Step function is defined as:



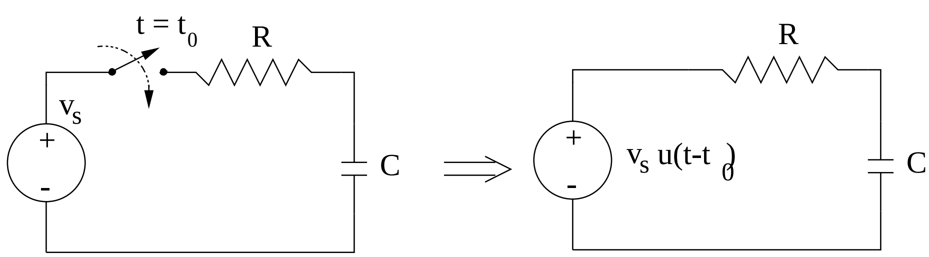
$$\begin{cases} u(t) = 0 & \text{for } t \leq 0^- \\ u(t) = 1 & \text{for } t \leq 0^+ \end{cases}$$

Defining $t' = t - t_0$, one can see that $u(t') = u(t - t_0)$ is:



$$\begin{cases} u(t - t_0) = 0 & \text{for } t \leq t_0^- \\ u(t - t_0) = 1 & \text{for } t \leq t_0^+ \end{cases}$$

Step functions are <u>one way</u> to illustrate switched circuit as is shown in the example below.



Step response of an RC circuit

Consider the RC circuit above. The switch closes at time $t = 0$ and the capacitor has an initial voltage of v_0. For $t > 0$, KVL results in $Ri_c + v_C = v_s$, or:

$$RC\frac{dv_C}{dt} + v_C = v_s$$

$$\text{I.C.:} \quad v_C(t = 0^+) = v_0$$

We have found the natural solution to RC circuit to be:

$$v_{C,n} = Ke^{-\frac{t}{\tau}} \quad \text{and} \quad \tau = RC$$

To find the forced response, $v_{C,f}$, we note that the RHS of the differential equation is a constant. Table of trial force functions on page 70 indicates that the forced response should also be a constant, $v_{C,f} = A$. Substituting for $v_{C,f}$ in the differential equation, we get:

$$RC\frac{dA}{dt} + A = v_s \quad \rightarrow \quad v_{C,f} = A = v_s$$

$$v_C = v_{C,n} + v_{C,f} = Ke^{-\frac{t}{\tau}} + v_s$$

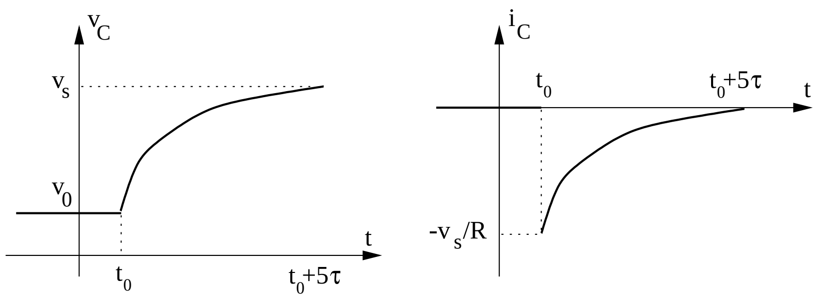
Constant of K is found from the initial condition: $v_C(t = 0^+) = v_0$:

$$v_C(t = 0^+) = v_0 = K e^{-\frac{0}{\tau}} + v_s = K + v_s \quad \rightarrow \quad K = v_0 - v_s$$

Thus, the capacitor voltage waveform is:

$$v_C(t) = (v_0 - v_s) e^{-\frac{t}{\tau}} + v_s$$

If we wait long enough, (mathematically: $t \rightarrow \infty$, practically: 5τ) the circuit will reach DC steady condition again, current in the capacitor becomes zero and its voltage reaches v_s as can be see either from the circuit or from the expression for $v_C(t)$.



If the switch was closed at time $t = t_0$ instead of time zero, the capacitor voltage waveform would be (let $t' = t - t_0$ and switch closing at $t' = 0$):

$$v_C(t) = (v_0 - v_s) e^{-\frac{t - t_0}{\tau}} + v_s$$

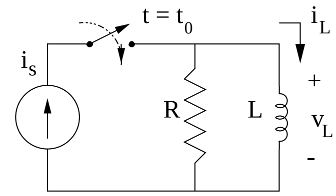
Since v_0 is the initial value of v_C and v_s is its final value , the above equation can be re-written as:

$$\begin{matrix} v_C \\ \text{at Time } t \end{matrix} = \left[\begin{matrix} \text{Initial Value of} \\ v_C \end{matrix} - \begin{matrix} \text{Final Value of} \\ v_C \end{matrix} \right] \times e^{-\frac{t - t_0}{\tau}} + \begin{matrix} \text{Final Value of} \\ v_C \end{matrix}$$

In fact, all voltages and currents in the circuit (also called "state variables") will have the same waveform:

$$\begin{matrix} \text{State Variable} \\ \text{at Time } t \end{matrix} = \left[\begin{matrix} \text{Initial Value of} \\ \text{State Variable} \end{matrix} - \begin{matrix} \text{Final Value of} \\ \text{State Variable} \end{matrix} \right] \times e^{-\frac{t - t_0}{\tau}} + \begin{matrix} \text{Final Value of} \\ \text{State Variable} \end{matrix}$$

Step response of an RL circuit



Consider the RL circuit shown. The switch closes at time $t = t_0$ and the inductor has an initial current of i_0. We can find the inductor current waveform following the procedure similar to one used for step response of RC circuits.

Alternatively, we can use the "state variable" formula identified above. Here the state variable of interest is i_L. The time constant of the circuit is $\tau = L/R$. The final value of the state variable is $i_L(t \to \infty)$ when the switch is closed and circuit has reached a DC steady state condition. Replacing the inductor with a short circuit, we find $i_L(t \to \infty) = i_s$. Substituting in the "state variable" formula above, we get

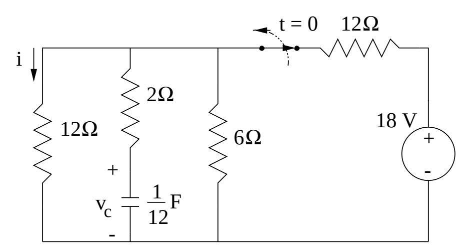
$$i_L(t) = (i_0 - i_s)e^{-\dfrac{t-t_0}{\tau}} + i_s$$

Procedure for Solving First-Order Circuits

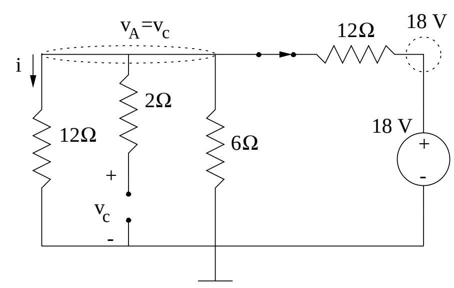
1. If the initial conditions are not given, use DC steady-state analysis to find the initial conditions (v_c and i_L)

2. Solve the time dependent circuit:
   a) Direct solution using KVL and KCL, node-voltage and mesh current methods, *etc.*
   b) Reduce the circuit to simple RC or RL circuits above and use the formulas.

Example 1: The circuit is in DC steady-state for $t < 0$. Find i for $t > 0$



As the initial conditions are not given, we need to solve the DC steady-state circuit for $t < 0$ first. We redraw the circuit at $t < 0$ (switch is closed) and replace the capacitor with an open circuit. We proceed with solving the circuit with node-voltage method. As the 2 Ω resistor does not carry any current, $v_A = v_C$. Then:

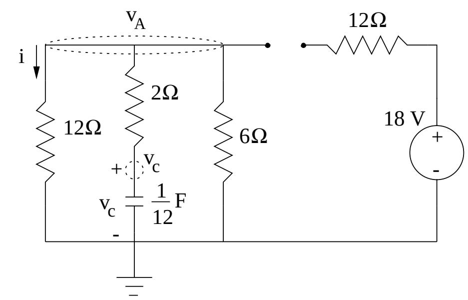


$$\text{KCL at } v_A: \quad \frac{v_C - 18}{12} + \frac{v_c}{6} + \frac{v_C}{12} = 0$$

$$v_C = 4.5 \text{ V} \qquad \text{for} \quad t < 0$$

No-jump condition leads to the initial condition for $t > 0$: $v_C(0^+) = v_C(0^-) = 4.5$ V. Note that although we like to find i, the initial condition is obtained for v_C as no-jump condition only applies to v_C and i may have a discontinuity at the switching time.

We now proceed to solve $t > 0$ circuit:



Method 1: Direct solution:

Both node-voltage and mesh-current methods lead to 2 equations. As we are interested in v_C, we proceed with node-voltage method:

$$\text{KCL at } v_A: \qquad \frac{v_A}{6} + \frac{v_A - v_C}{2} + \frac{v_A}{12} = 0$$

$$2v_A + 6v_A - 6v_C + v_A = 0 \quad \rightarrow \quad v_A = \frac{2}{3}\, v_C$$

$$\text{KCL at } v_C: \qquad \frac{v_C - v_A}{2} + i_C = 0 \quad \rightarrow \quad \frac{v_C - v_A}{2} + \frac{1}{12}\frac{dv_C}{dt} = 0$$

where we substituted for i_C from the capacitor i-v equation. The above are two equations in our two node-voltages v_A and v_C. Substituting for v_A from first into the second, we get:

$$6v_C - 6\left(\frac{2}{3}\, v_C\right) + \frac{dv_C}{dt} = 0$$

$$\frac{dv_C}{dt} + 2v_C = 0 \qquad \text{and} \qquad v_C(0^+) = 4.5 \text{ V}$$

As the RHS of the differential equation is zero, solution consists only of the natural solution. Using the trial function of Ke^{st}, we find:

$$sKe^{st} + 2Ke^{st} = 0 \quad \rightarrow \quad s = -2$$

$$v_C(t) = Ke^{-2t}$$

and K is found from the initial condition:

$$v_C(0^+) = 4.5 = Ke^{-2\times 0} \quad \rightarrow \quad K = 4.5 \quad \rightarrow \quad v_C(t) = 4.5e^{-2t} \quad \text{(V)}$$

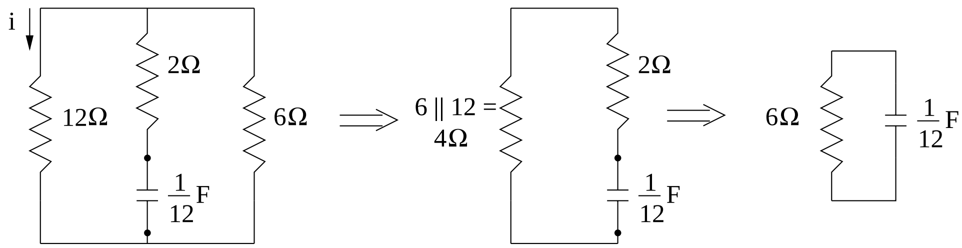
We can now calculate i from our node voltage equations:

$$v_A(t) = \frac{2}{3}\, v_C(t) = 3e^{-2t} \quad \text{(V)}$$

$$i(t) = \frac{v_A(t)}{12} = 0.25e^{-2t} \quad \text{(A)}$$

Method 2: Reduction to Thevenin form:

In this method, we reduce the circuit into a simple RC circuit by separating the capacitor from the circuit and finding Thevenin equivalent of the remaining two-terminal subcircuit:



We have solved this circuit before and the solution is:

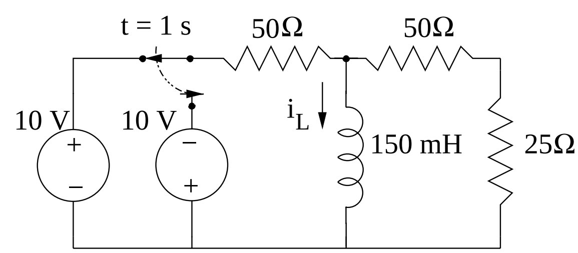
$$v_C(t) = v_C(t = t_0^+)\, e^{-\dfrac{t - t_0}{\tau}} \qquad \text{with} \qquad \tau = RC$$

$$t_0 = 0 \qquad \text{and} \qquad v_C(t = t_0^+) = v_C(0^+) = 4.5 \qquad \text{and} \qquad \tau = RC = \frac{6}{12} = 0.5$$

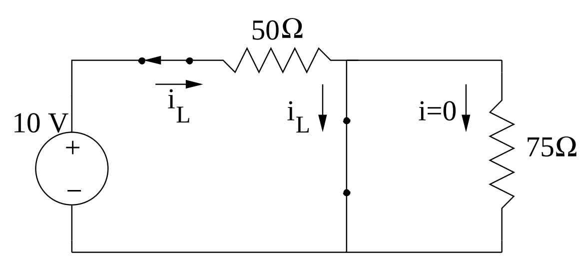
$$v_C(t) = 4.5 e^{-2t} \quad \text{(V)}$$

We now need to go back to the original circuit to calculate i, for example, by writing the node-voltage equations and use v_C to find the other parameters as was done above.

Example 2: The circuit is in DC steady-state for $t < 1$ s. Find i_L for $t > 1$



We redraw the circuit at $t < 1$ (switch is in the upper position) and replace the inductor with a short circuit. We also replace the 50 and 25 Ω resistors in series with a 75 Ω resistor. As the 75 Ω resistor is in parallel with a short circuit, it will carry a current of zero. Then, by KCL, the current in the 50 Ω resistor will be i_L. Then,

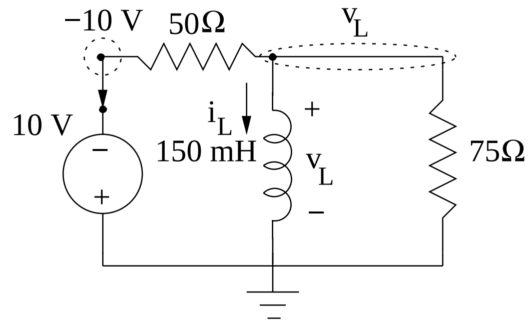


$$\text{KVL:} \qquad -10 + 50 i_L = 0 \quad \rightarrow \quad i_L = 0.2 \text{ A}$$

No-jump condition leads to the initial condition for $t > 1$: $i_L(t = 1^+) = i_L(t = 1^-) = 0.2$ A.

We now proceed to solve $t > 0$ circuit:

Method 1: Direct solution:



Node-voltage method leads to one equation as opposed to mesh-current method that leads to 2 equations. So, we proceed with node-voltage method:

$$\text{KCL at } v_L: \quad \frac{v_L}{75} + i_L + \frac{v_L - (-10)}{50} = 0$$

$$2v_L + 150 i_L + 3v_L + 30 = 0 \quad \rightarrow \quad 5v_L + 150 i_L = -30$$

$$i\text{-}v \text{ Eq.:} \quad v_L = 150 \times 10^{-3} \frac{di_L}{dt}$$

$$5 \times 150 \times 10^{-3} \frac{di_L}{dt} + 150 i_L = -30$$

$$5 \times 10^{-3} \frac{di_L}{dt} + i_L = -0.2 \quad \text{and} \quad i_L(t = 1^+) = 0.2 \text{ A.}$$

As the RHS of the differential equation is not zero, we need to find both the natural and forced solutions. The natural solution can be found by using trial function of $i_{L,n} = Ke^{st}$:

$$5 \times 10^{-3} sKe^{st} + Ke^{st} = 0 \quad \rightarrow \quad s = -200$$

$$i_{L,n}(t) = Ke^{-200t}$$

To find $i_{L,f}$, we note that the RHS of the differential equation is a constant. Using the Table of trial functions for forced solution on page 70, we find $i_{L,f} = A$. Substituting in the differential equation, we get:

$$5 \times 10^{-3} \frac{dA}{dt} + A = -0.2 \quad \rightarrow \quad i_{L,f} = A = -0.2$$

$$i_L(t) = i_{L,n} + i_{L,f} = Ke^{-200t} - 0.2$$

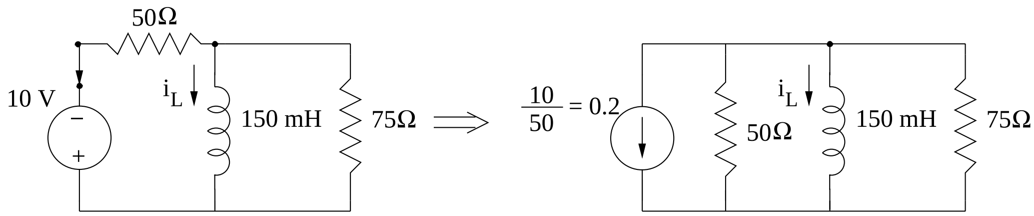
Constant K is found from the initial conditions:

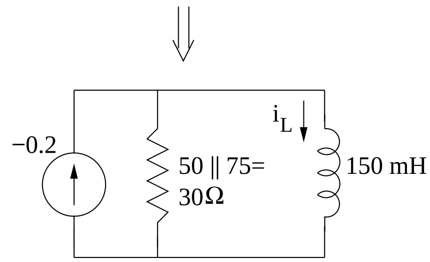
$$i_L(t = 1^+) = 0.2 = Ke^{-200} - 0.2 \quad \rightarrow \quad K = 0.4e^{+200}$$

$$i_L(t) = 0.4e^{-200(t-1)} - 0.2 \quad \text{(A)}$$

Method 2: Reduction to Thevenin form:

In this method, we reduce the circuit into a simple RL circuit by separating the capacitor from the circuit and finding Thevenin equivalent of the remaining two-terminal subcircuit:





We have solve this circuit before and the solution is:

$$i_L(t) = (i_0 - i_s)e^{-\dfrac{t-t_0}{\tau}} + i_s$$

$$t_0 = 1$$

$$i_0 = i_L(t = t_0^+) = +0.2 \quad \text{and} \quad i_s = -0.2$$

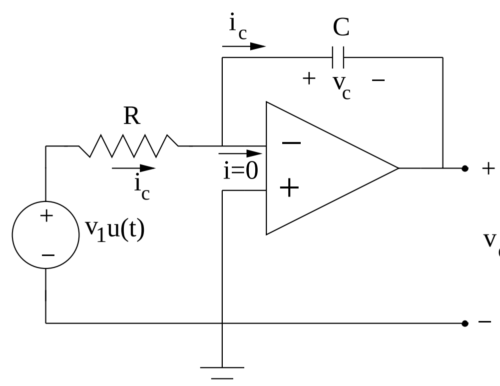
$$\tau = \frac{L}{R} = \frac{150 \times 10^{-3}}{30} = 5 \times 10^{-3}$$

$$i_L(t) = [0.2 - (-0.2)]e^{-200(t-1)} + (-0.2)$$

$$i_L(t) = 0.4e^{-200(t-1)} - 0.2 \quad \text{(A)}$$

Example 3: Integrator Find v_o if $v_C(t = 0) = 0$.

We replace the OpAmp with its circuit model. As the current flowing into the OpAmp is zero, the current in the resistor is the same as i_C. We also note the connection between output and inverting input terminal so negative feedback exists:



$$\begin{aligned} \text{Negative Feedback:} \quad & v_n = v_p = 0 \\ \text{Ohm's Law:} \quad & Ri_C = v_1 - v_n = v_1 \\ \text{Capacitor } i\text{-}v\text{:} \quad & i_c = C\frac{dv_C}{dt} = C\frac{d(v_n - v_o)}{dt} = -C\frac{dv_o}{dt} \end{aligned}$$

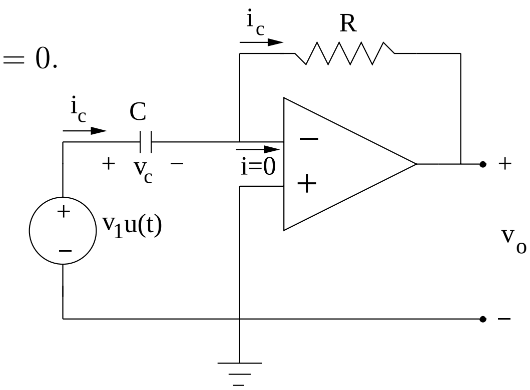
Substituting for i_C from Ohm's law into capacitor i-v equation, and integrating the resulting equation we get:

$$\frac{v_1}{R} = -C\frac{dv_o}{dt} \quad \rightarrow \quad \int_0^t \frac{dv_o}{dt'}dt' = -\frac{1}{RC}\int_0^t v_1(t')dt'$$

$$v_o(t) = -\frac{1}{RC}\int_0^t v_1(t')dt'$$

since $v_C(t = 0) = v_o(t = 0) = 0$. As can be seen, this is an integrator circuit–the output voltage is proportional to the integral of the input voltage waveform.

Example 4: Differentiator: Find v_o if $v_C(t=0)=0$.



We replace the OpAmp with its circuit model. As the current flowing into the OpAmp is zero, the current in the resistor is the same as i_C. We also note the connection between output and inverting input terminal so negative feedback exists:

$$
\begin{aligned}
\text{Negative Feedback:} \quad & v_n = v_p = 0 \\
\text{Ohm's Law:} \quad & R i_C = v_n - v_o = -v_o \\
\text{Capacitor } i\text{-}v: \quad & i_c = C\frac{dv_C}{dt} = C\frac{d(v_1 - v_n)}{dt} = C\frac{dv_1}{dt}
\end{aligned}
$$

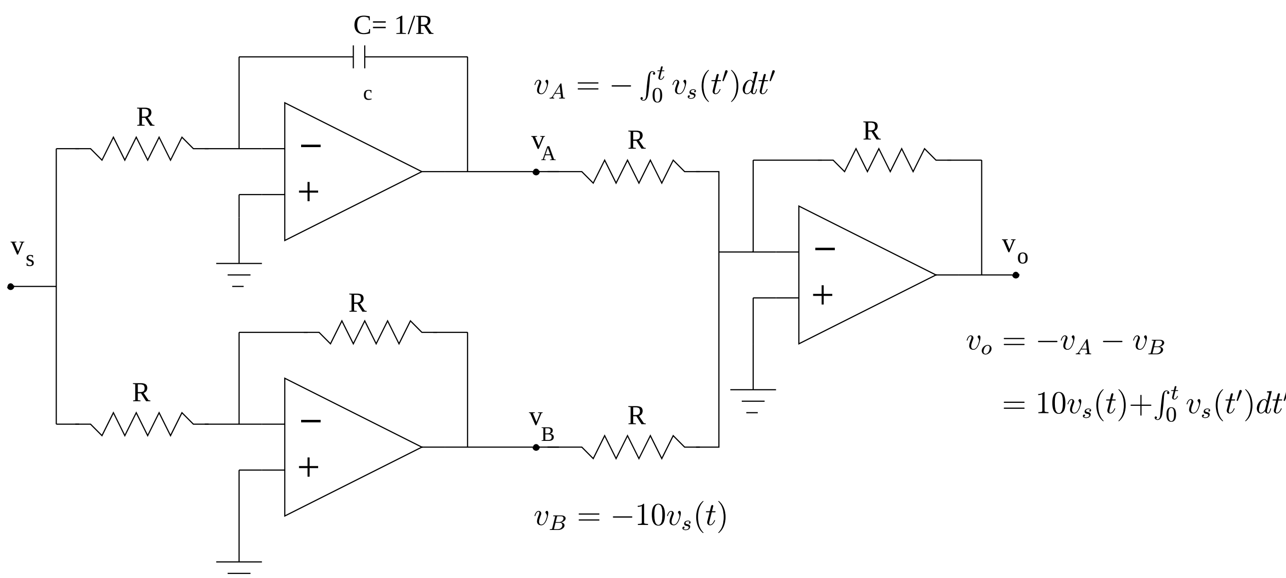
Substituting for i_C from Ohm's law into capacitor i-v equation, we get:

$$
-\frac{v_o}{R} = C\frac{dv_1}{dt} \quad \rightarrow \quad v_o(t) = -RC\frac{dv_1}{dt}
$$

As can be seen, this is a differentiator circuit–the output voltage is proportional to the derivative of the input voltage waveform.

The above two circuits, the integrator and the differentiator, together with inverting and non-inverting summers are the building block of **analog computers.**

Example: Design an OpAmp circuit to find $v_o(t) = 10v_s(t) + \int_0^t v_s(t')dt'$.

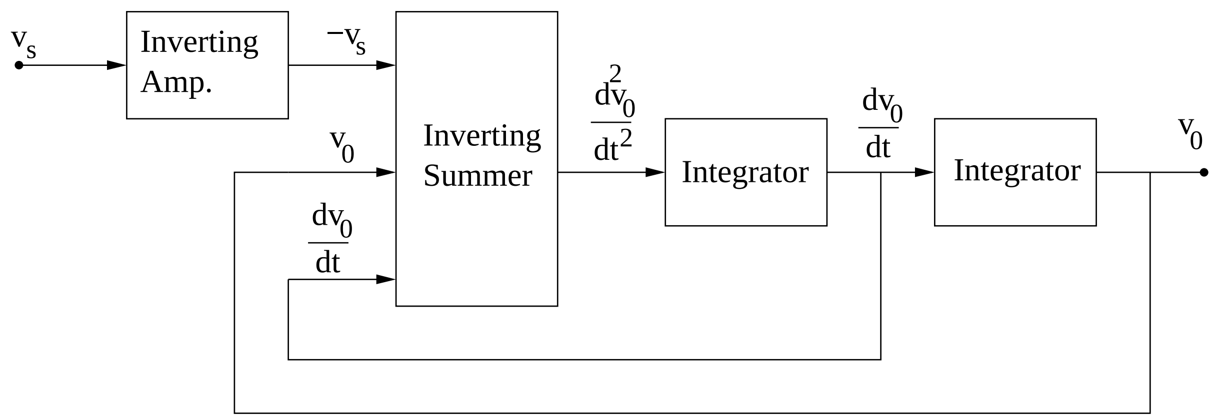


Example: Design an OpAmp circuit which solves the differential equation:
$\dfrac{d^2v_o}{dt^2} + 2\dfrac{dv_o}{dt} + v_o = v_s(t).$

Rewrite the equation in the form:

$$\frac{d^2v_o}{dt^2} = -2\frac{dv_o}{dt} - v_o + v_s(t)$$

The block diagram of the circuit is:



and the circuit itself is: