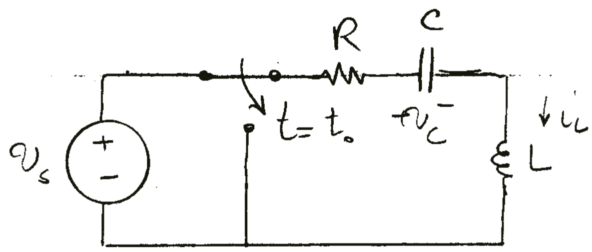
SECOND ORDER CIRCUITS

Second order circuits contain <u>two</u> storage elements (2 Capacitor or 2 inductor or one of each)

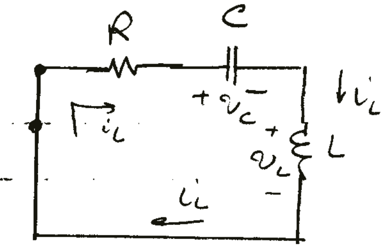
- The solution to this circuits always results in 2^{nd} order differential equation & requires <u>two</u> initial Conditions
- Solution Method is similar to 1^{st} order Circuits

SERIES RLC Circuits, Natural Response



DC steady-state analysis will determine the inital Conditions:

$$v_c(t = t_0^+) \quad \& \quad i_L(t = t_0^+)$$



$t > t_0$

KVL:
$$\begin{cases} Ri_L + v_C + v_L = 0 \\ v_L = L\dfrac{di_L}{dt} \\ i_C = i_L = C\dfrac{dv_C}{dt} \end{cases}$$

$$\begin{cases} L\dfrac{di_L}{dt} + Ri_L + v_C = 0 \\ i_L = C\dfrac{dv_C}{dt} \end{cases}$$

Substitute for i_L in the first equation noting $\dfrac{di_L}{dt} = C\dfrac{d^2v_C}{dt^2}$

$$LC\frac{d^2v_C}{dt} + RC\frac{dv_C}{dt} + v_C = 0$$

above is a 2^{nd} order differential equation for v_C. We need two initial conditions for v_C, typically

$$v_C(t=t_o^+) \quad \& \quad \left.\frac{dv_C}{dt}\right|_{t=t_o^+}$$

$v_C(t=t_o^+)$ is usually found from DC steady state analysis. In order to find $\left.\frac{dv_C}{dt}\right|_{t=t_o^+}$, look at equations governing the circuit & choose one which include both i_L & $\frac{dv_C}{dt}$

$$i_L = C\frac{dv_C}{dt}$$

Since the above equation is correct at all times $t>t_o$, thus

$$i_L(t=t_o^+) = C\left.\frac{dv_C}{dt}\right|_{t=t_o^+} \Rightarrow \underbrace{\left.\frac{dv_C}{dt}\right|_{t=t_o^+} = \frac{i_L(t=t_o^+)}{C}}_{2^{nd}\text{ inital condition}}$$

Note that we could have, alternatively, found a 2^{nd} order differential equation in i_L by differentiating the first equation & substituting for $\frac{dv_C}{dt}$

$$L\frac{di_L}{dt} + Ri_L + v_C = 0$$

$$L\frac{d^2i_L}{dt^2} + R\frac{di_L}{dt} + \frac{dv_C}{dt} = 0 \qquad \frac{dv_C}{dt} = \frac{i_L}{C}$$

$$\boxed{LC\frac{d^2i_L}{dt^2} + RC\frac{di_L}{dt} + i_L = 0}$$

The inital conditions for this equation is also found as before
$i_L(t=t_o^+)$ from DC steady state

$$L\frac{di_L}{dt} + Ri_L + v_C = 0 \Rightarrow \left.\frac{di_L}{dt}\right|_{t=t_o^+} = -\frac{R}{L}i_L(t=t_o^+) + \frac{1}{L}v_C(t=t_o^-)$$

Solution: try $v_c = Ke^{st} \rightarrow \frac{dv_c}{dt} = Kse^{st}$, $\frac{d^2v_c}{dt^2} = Ks^2e^{st}$

Thus: $LC(Ks^2e^{st}) + RC(Kse^{st}) + Ke^{st} = 0$

or $LCs^2 + RCs + 1 = 0$ ⇐ Characteristics Equation

All 2^{nd} order differential equation result in a 2^{nd} order characteristics equation ⇒ two roots s_1 & s_2

$$v_c = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

For special case of $s_1 = s_2$ ⇒ $v_c = K_1 t e^{s_1 t} + K_2 e^{s_1 t}$

In order to review the general behavior of 2^{nd} order circuits, rewrite the characteristics equation as

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

Or in general $s^2 + 2\alpha s + \omega_0^2 = 0 \Rightarrow s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

(for above circuit, $\alpha = \frac{R}{2L}$, $\omega_0^2 = \frac{1}{LC}$) α: Neper frequency, ω_0: Resonant radian freq.

Depending on values of α & ω_0^2, we can have 1) two real roots, 2) two complex roots, 3) two purely imaginary roots, 4) a 2^{nd} order root ($s_1 = s_2$)

Case I $\alpha = 0$, $\omega_0^2 > 0$ ($R = 0$ in series RLC)

$s^2 + \omega_0^2 = 0 \Rightarrow s_1 = +j\omega_0 \quad s_2 = -j\omega_0 \quad (j^2 = -1)$

Then $v_c = K_1 e^{j\omega_0 t} + K_2 e^{-j\omega_0 t} = K \cos(\omega_0 t + \varphi_0)$

by Euler's formula

K & Φ_0 are constants of integration & can be found from the initial conditions.

For our example

$$\begin{cases} v_C(t=t_0^+) = v_s = K\cos(\omega_0 t_0 + \Phi_0) \\ \left.\dfrac{dv_C}{dt}\right|_{t=t_0^+} = 0 = -K\omega_0 \sin(\omega_0 t_0 + \Phi_0) \Rightarrow \omega_0 t_0 + \Phi_0 = 0 \end{cases}$$

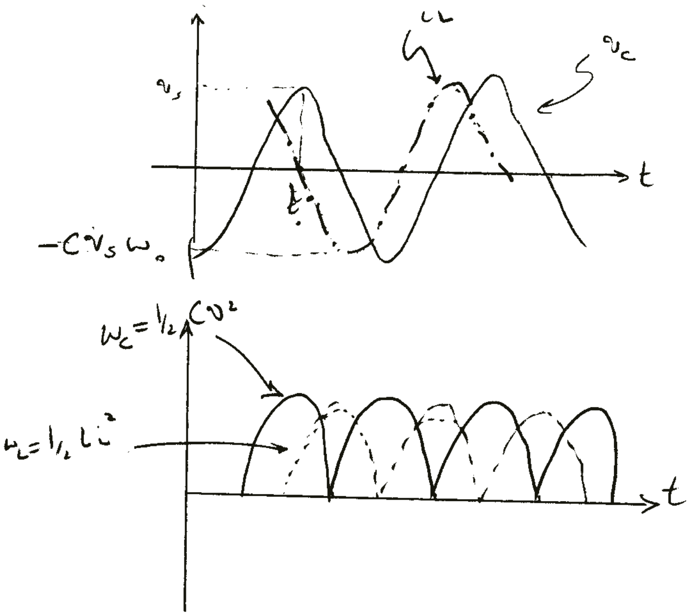
$$\Phi_0 = -\omega_0 t_0$$

$$v_s = K\cos(\omega_0 t_0 - \omega_0 t_0) = K \rightarrow K = v_s$$

Solution

$$\begin{cases} v_C = v_s \cos[\omega_0 (t - t_0)] \\ i_L = C\dfrac{dv_C}{dt} = -C v_s \omega_0 \sin[\omega_0(t-t_0)] \end{cases} \qquad \omega_0 = \sqrt{\frac{1}{LC}}$$

This is harmonic Oscillator



Note at the $t = t_0$, Capacitor is fully charged ($v_C = v_s$) & inductor is fully discharged ($i_L = 0$). As the switch is moved to position B, Capacitor discharges (v_C decrease) & inductors charges up (i_L increases). Once Capacitor is fully discharged ($v_C = 0$), the inductor is fully charged & Then it starts to discharge, charging up the capacitor. This is very similar to the motion of a pendulum.

$$w_C = \frac{1}{2}Cv_C^2 = \frac{C}{2}v_s^2 \cos^2[\omega_0(t-t_0)]$$

$$w_L = \frac{1}{2}Li_L^2 = \frac{1}{2}LC^2 v_s^2 \omega_0^2 \sin^2[\omega_0(t-t_0)] = \frac{1}{2}Cv_s^2 \sin^2[\omega_0(t-t_0)]$$

$$\omega_0^2 = \frac{1}{LC}$$

$$w_{total} = w_C + w_L = \frac{C}{2}v_s^2 = w_C(t = t_0^+)$$

Case II : $\alpha \neq 0$ but α small $(\alpha^2 < \omega_0^2)$

Underdamped

$$\left(\frac{R}{2L}\right)^2 < \frac{1}{LC} \Rightarrow R^2 < \frac{L}{2C}$$

When a resistor is present, the total stored energy is not constant any more. During each oscillation, a part of stored energy is dissipated by the resistor. Therefore, the solution should be like of a harmonic oscillator but with amplitude of oscillation being reduced in time (similar to a real pendulum, in which friction would stop the pendulum eventually).

$$s^2 + 2\alpha s + \omega_0^2 = 0 \Rightarrow s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

Define $\omega_d^2 = \omega_0^2 - \alpha^2 > 0 \Rightarrow s = -\alpha \pm j\omega_d$

Solution $v_c(t) = K_1 e^{(-\alpha + j\omega_d)t} + K_2 e^{(-\alpha - j\omega_d t)}$

$$= e^{-\alpha t}\left[K_1 e^{j\omega_d t} + K_2 e^{-j\omega_d t}\right]$$

$$v_c = K e^{-\alpha t} \cos(\omega_d t + \phi_0)$$ using Euler's formula as before

Again K & ϕ_0 are found from our initial condition

① $\left.\frac{dv_c}{dt}\right|_{t=t_0^+} = K\left\{-\alpha \cos(\omega_d t_0 + \phi_0) e^{-\alpha t_0} - \omega_d \sin(\omega_d t_0 + \phi_0) e^{-\alpha t_0}\right\} = 0$

$$\tan(\omega_d t_0 + \phi_0) = -\frac{\alpha}{\omega_d} \Rightarrow \phi_0 + t_0\omega_d = \tan^{-1}\left(-\frac{\alpha}{\omega_d}\right)$$

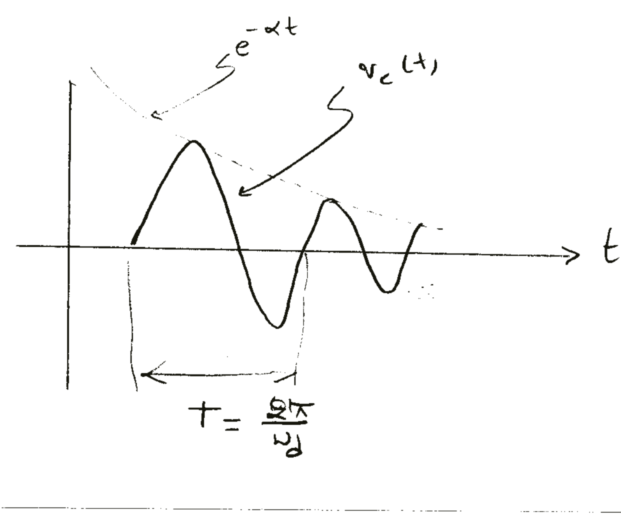
② $v_s = v_c(t = t_0^+) = K e^{-\alpha t_0} \cos(\omega_d t_0 + \phi_0) \Rightarrow K$

Note, because $\omega_d = \sqrt{\omega_0^2 - \alpha^2} < \omega_0$

The frequency of oscillations become slower!

ω_d : damped frequency

α : damping coefficient.



Case III , $\alpha \neq 0$ & $\alpha^2 = \omega_0^2$ Critically damped

$$s^2 + 2\alpha s + \omega_0^2 = 0 \Rightarrow s^2 + 2\alpha s + \alpha^2 = 0 \Rightarrow s_1 = s_2 = -\alpha$$

Solution $\quad v_C = K_1 e^{-\alpha t} + K_2 t e^{-\alpha t}$

I.C. $\quad v_C(t = t_0^+) = v_s \Rightarrow K_1 e^{-\alpha t_0} + K_2 t_0 e^{-\alpha t_0} = v_s \Rightarrow K_1 + K_2 t_0 = v_s e^{\alpha t_0}$

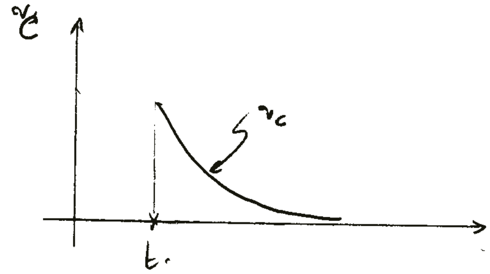
$$\left.\frac{dv_C}{dt}\right|_{t=t_0^+} = 0 = \left[-\alpha K_1 e^{-\alpha t} + K_2 e^{-\alpha t} - \alpha K_2 t e^{-\alpha t}\right]\Big|_{t=t_0^+}$$

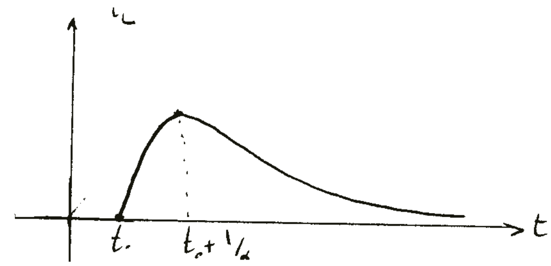
$$-\alpha K_1 e^{-\alpha t_0} + K_2 e^{-\alpha t_0} - \alpha K_2 t_0 e^{-\alpha t_0} = 0 \Rightarrow \alpha K_1 + K_2 + \alpha K_2 t_0 = 0$$

$\therefore \quad K_2 = +\alpha (K_1 + K_2 t_0) = +\alpha v_s e^{\alpha t_0}$

$K_1 = v_s e^{\alpha t_0} - \alpha t_0 v_s e^{\alpha t_0} = (1 - \alpha t_0) v_s e^{\alpha t_0}$

$$v_C = (1 - \alpha t_0) v_s e^{-\alpha(t - t_0)} + \alpha v_s t e^{-\alpha(t - t_0)}$$





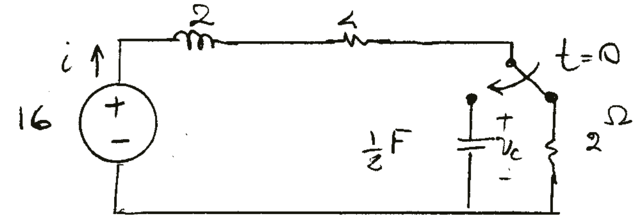
Case IV $\quad \alpha^2 > \omega_0^2 \quad$ Overdamped (α can be zero)

$$s^2 + 2\alpha s + \omega_0^2 = 0 \qquad s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$v_C = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

Again K_1 & K_2 can be found from I.C. . The waveforms are similar to critically damped case. the Critically damped case, however, reaches steady state conditions the fastest.

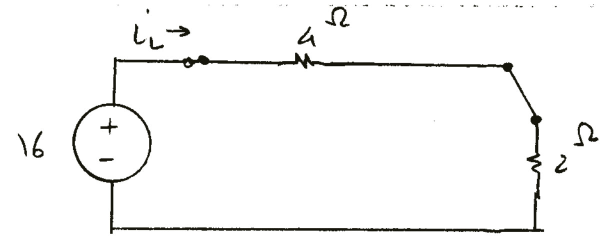
Example: Circuit below is in DC steady state for $t < 0^s$.
Find $i(t)$ after the switch has been moved.



$t<0$ (DC steady state)

$v_c = 0$

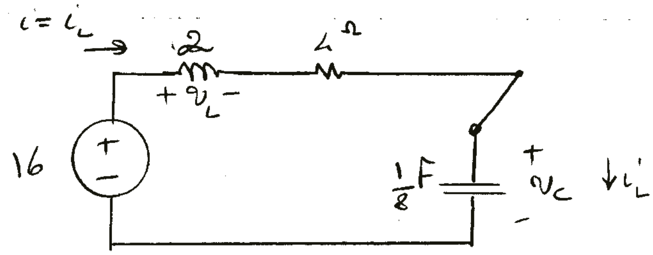
KVL: $i_L = \frac{16}{6} = \frac{8}{3}$ A



Thus $i_L(t=0^+) = \frac{8}{3}$ A

$v_c(t=0^+) = 0$

$t>0$



KVL: $v_L + 4i_L + v_c - 16 = 0$

Element laws: $v_L = L\frac{di_L}{dt} = 2\frac{di_L}{dt}$

$i_c = i_L = C\frac{dv_c}{dt} = \frac{1}{8}\frac{dv_c}{dt}$

Thus
$$\begin{cases} 2\frac{di_L}{dt} + 4i_L + v_c = 16 \\ i_L = \frac{1}{8}\frac{dv_c}{dt} \end{cases}$$

Substituting for i_L in the first equation will result in a 2^{nd} order differential equation. Alternatively, substituting for v_c from 1^{st} equation into the 2^{nd} equation will result in a 2^{nd} order differential equation for i_L

$$8i_L = \frac{dv_c}{dt} = \frac{d}{dt}\left[16 - 2\frac{di_L}{dt} - 4i_L\right]$$

$$2\frac{d^2i_L}{dt^2} + 4\frac{di_L}{dt} + 8i_L = 0$$

$$\frac{d^2 i_L}{dt^2} + 2\frac{di_L}{dt} + 4 i_L = 0 \quad \leftarrow$$

initial Conditions: $\quad i_L(t=0^+) = \frac{8}{3} A$

To find $\left.\frac{di_L}{dt}\right|_{t=0^+}$, evaluate KVL at $t=$

$$2\left.\frac{di_L}{dt}\right|_{t=0^+} + 4 i_L(t=0^+) + v_C(t=0^+) = 16$$

$$2\left.\frac{di_L}{dt}\right|_{t=0^+} + \frac{32}{3} + 0 = 16 \Rightarrow \left.\frac{di_L}{dt}\right|_{t=0^+} = \frac{8}{3}$$

Solution (note no forced response). Try $i_L = Ke^{st}$

$$Ks^2 e^{st} + 2Ks e^{st} + 4Ke^{st} = 0$$

$$s^2 + 2s + 4 = 0 \quad \leftarrow \text{Characteristic Eq.}$$

Note $\alpha = 1$, $\omega_0^2 = 4 \Rightarrow$ underdamped solution $\quad (\omega_d^2 = 3)$

$$s = -1 \pm \sqrt{1-4} = -1 \pm j\sqrt{3}$$

Thus $\quad i_L = e^{-t}\left[K_1 \cos(\sqrt{3}t) + K_2 \sin(\sqrt{3}t)\right]$

To find K_1 & K_2, use initial conditions

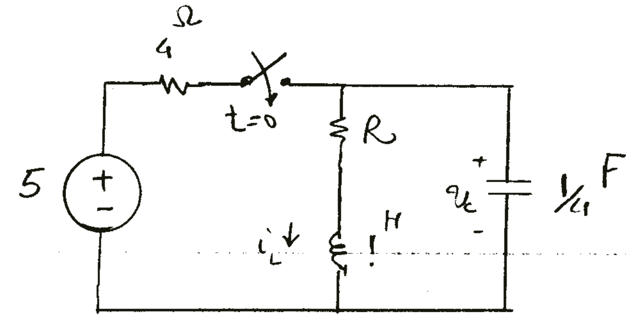
$$\begin{cases} i_L(t=0^+) = K_1 = \frac{8}{3} \\ \left.\frac{di_L}{dt}\right|_{t=0^+} = -e^{-t}\left[K_1 \cos(\sqrt{3}t) + K_2 \sin(\sqrt{3}t)\right] + e^{-t}\left[-\sqrt{3}K_1 \sin(\sqrt{3}t) + \sqrt{3}K_2 \cos(\sqrt{3}t)\right] \end{cases}$$

$$= -K_1 + \sqrt{3}K_2 = \frac{8}{3}$$

$$K_1 = \frac{8}{3}, \quad K_2 = \frac{16}{3\sqrt{3}} \Rightarrow i_L(t) = e^{-t}\left[\frac{8}{3}\cos(\sqrt{3}t) + \frac{16}{3\sqrt{3}}\sin(\sqrt{3}t)\right]$$

$$i_L(t) = e^{-t}\left[2.67\cos(1.73t) + 3.08\sin(1.73t)\right]$$

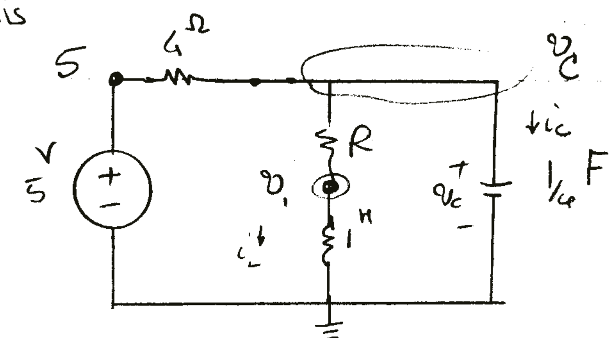
Example: a) For what value of R, the circuit below is critically damped
b) find $v_c(t)$ for $t>0$



for $t \le 0^-$, circuit is in DC steady state with $i_L=0$, $v_c=0$

Thus $i_L(t=0^+)=0$, $v_c(t=0^+)=0$

for $t>0$, use Nodal analysis



Nodes 1:
$$\frac{v_1-v_C}{R}+i_L=0$$

Node C: $$i_c+\frac{v_c-v_1}{R}+\frac{v_c-5}{4}=0$$

Aux: $$i_c=C\frac{dv_c}{dt}=\frac{1}{4}\frac{dv_c}{dt}$$

$$v_1=L\frac{di_L}{dt}=\frac{di_L}{dt}$$

Use the two aux. equation for i_c & i_L to substitute in the nodal equations.

Node 1: $$v_1-v_c+Ri_L=0$$

Differentiate ⇒ $\frac{dv_1}{dt}-\frac{dv_c}{dt}+R\frac{di_L}{dt}=0$ ⇒ substitute $v_1=\frac{di_L}{dt}$

$$\frac{dv_1}{dt}-\frac{dv_c}{dt}+Rv_1=0 \leftarrow$$

Node C: $$4Ri_c+4v_c-4v_1+Rv_c-5R=0$$

substitute for i_c → $R\frac{dv_c}{dt}+(4+R)v_c-4v_1=5R$

$$\begin{cases} \dfrac{dv_1}{dt} - \dfrac{dv_c}{dt} + Rv_1 = 0 \\ R\dfrac{dv_c}{dt} + (4+R)\,v_c - 4v_1 = 5R \end{cases}$$

In order to reduce this to one 2nd or equation :

$$4v_1 = +R\frac{dv_c}{dt} + (4+R)\,v_c - 5R$$

differentiate

$$4\frac{dv_1}{dt} = +R\frac{d^2v_c}{dt^2} + (4+R)\frac{dv_c}{dt}$$

1st equation $\Rightarrow$ $4\dfrac{dv_1}{dt} - 4\dfrac{dv_c}{dt} + 4Rv_1 = 0$

$$+R\frac{d^2v_c}{dt^2} + (4+R)\frac{dv_c}{dt} - 4\frac{dv_c}{dt} + R\left[+R\frac{dv_c}{dt} + (4+R)\,v_c - 5R\right] = 0$$

$$\cancel{R}\frac{d^2v_c}{dt^2} + R\frac{dv_c}{dt} + \cancel{R}\left[\qquad\right] = 0$$

$$\frac{d^2v_c}{dt^2} + \frac{dv_c}{dt}(1+R) + (4+R)\,v_c = 5R$$

Substitute $v_c = Ke^{st}$ in the homogenous equation to find the characteristic equation:

$$s^2 + (1+R)s + (4+R) = 0$$

$$\omega_0^2 = 4+R \qquad \alpha = \frac{1+R}{2}$$

For system to be critically damped $\alpha^2 = \omega_0^2$

$$4+R = \frac{(1+R)^2}{4} \Rightarrow 16 + 4R = 1 + 2R + R^2$$

$$R^2 - 2R - 15 = 0$$

Solution $R = 5$. $R = -3$ unphysical

$$R = 5$$

thus : $$\frac{d^2v_c}{dt^2} + 6\frac{dv_c}{dt} + 9v_c = 25$$

initial Conditions $\underline{v_c(t=0^+)=0}$

To find the 2nd initial condition ($\frac{dv_c}{dt}\Big|_{t=0^+}$) , we note

Node c ⇒ $$R\frac{dv_c}{dt} + (4+R)v_c - 4v_1 = 5R$$

Node 1 ⇒ $$v_1 = v_c + Ri_L$$

Thus (with $R=5$) ⇒ $$5\frac{dv_c}{dt} + 9v_c - 4(v_c + 5i_L) = 25$$

evaluate at $t=0^+$ $$5\frac{dv_c}{dt}\Big|_{t=0^+} + 5\cancel{v_c(t=0^+)} - 20\cancel{i_L(t=0^+)} = 25$$

$$\frac{dv_c}{dt}\Big|_{t=0^+} = \underline{5}$$

Solution: $$v_c = v_{c,n} + v_{c,f}$$

to find $v_{c,n}$ ⇒ characteristic equation: $s^2+6s+9=0$, $s_1=s_2=-3$

$$v_{c,n} = K_1e^{-3t} + K_2te^{-3t}$$

$v_{c,f}$, using Table on page (88) ⇒ trial function $v_{c,f}=A$

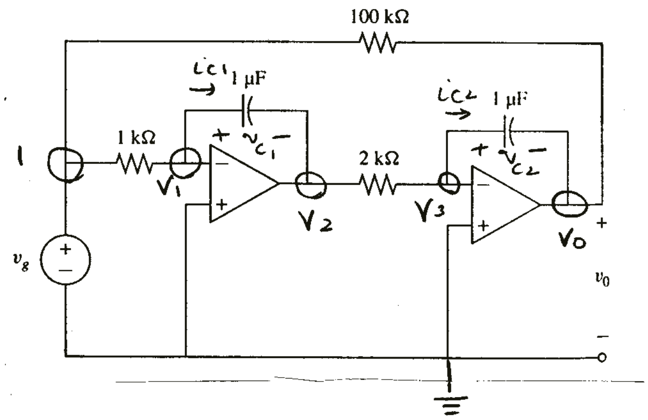
$$0+6\times 0+9A=25 \Rightarrow A = 25/9$$

Thus $$v_c(t) = K_1e^{-3t} + K_2te^{-3t} + 25/9$$

Using initial Conditions, we find $K_1 = -25/9$, $K_2 = \frac{-10}{3}$

$$v_c(t) = -2.778\,e^{-3t} - 3.33\,te^{-3t} + 2.778$$

Example: Find step Response.



step response:

$t<0$ $\quad v_g=0$, DC steady state

$v_{c_1}=v_{c2}=0$

$t>0 \quad v_g=1$

Use nodal analysis

I.C. $\quad v_{c_1}(t=0^+) = V_1(t=0^+) - V_2(t=0^+)=0$

$v_{c2}(t=0^+) = V_o(t=0^+) - V_3(t=0^+)=0$

Node 1: $\quad \dfrac{V_1-1}{10^3} + i_{c_1} = 0$

Node 2 $\Rightarrow$ Op Amp output $\Rightarrow v_{d_1}=0 \Rightarrow V_1=0$

Node 3 $\Rightarrow \quad \dfrac{V_3-V_2}{2\times10^3} + i_{c2}=0$

Node 4 $\Rightarrow$ Op Amp output $\Rightarrow v_{d2}=0 \Rightarrow V_3=0$

Aux Eqn. $\quad i_{c_1} = 10^{-6}\dfrac{dv_{c1}}{dt} = 10^{-6}\dfrac{d}{dt}(V_1-V_2)$

$i_{c2} = 10^{-6}\dfrac{dv_{c2}}{dt} = 10^{-6}\dfrac{d}{dt}(V_o-V_3)$

Thus:
$$\begin{cases} V_1=0 \\ V_3=0 \\ i_{c_1} = +10^{-3} = 10^{-6}\dfrac{d}{dt}(-V_2) \\ i_{c2} = 0.5\times10^{-3}\,V_2 = 10^{-6}\dfrac{d}{dt}(V_o) \end{cases}$$

$$\begin{cases} \dfrac{dV_2}{dt} = -10^{+3} \\ \dfrac{dV_o}{dt} = 0.5\times10^{3}\,V_2 \end{cases}$$

I.C. $\quad V_2(t=0^+)=0, \quad V_o(t=0^+)=0$

Solution:

★ General Method.

differentiate the 2^{nd} equation & substitute for $\frac{dV_2}{dt}$

$$\frac{d^2 V_o}{dt^2} = 0.5\times10^{3}\frac{dV_2}{dt} = -0.5\times10^{6}$$

$$\text{I.C.} \quad V_o(t=0^+)=0 \qquad \left.\frac{dV_o}{dt}\right|_{t=0^+} = 0.5\times10^{-3}\, V_2(t=0^+) = 0$$

Thus integrating

$$\frac{dV_o}{dt} = -0.5\times10^{6}\,t + K_1$$

$$V_o = -0.25\times10^{6}\,t^2 + K_1 t + K_2$$

using the intial Conditions $\Rightarrow$ $K_1=0$, $K_2=0$ $\Rightarrow$ $V_o = -0.25\times10^{6}t^2$

★For Cascade op Amp circuit, the nodal differential equation usually are in the form that Can be directly integrated,

$$\frac{dV_2}{dt} = -10^{3} \Rightarrow \quad V_2(t) = -10^{3}t + K_1$$

$$V_2(t=0^+)=0 \Rightarrow K_1=0 \Rightarrow V_2(t) = -10^{+3}t$$

$$\frac{dV_o}{dt} = 0.5\times10^{3}\,V_2 = -0.5\times10^{6}\,t$$

$$V_o(t) = -0.25\times10^{6}\,t^2 + K_2$$

$$V_o(t=0^+)=0 \Rightarrow K_2=0$$

$$V_o(t) = -0.25\times10^{6}\,t^2$$

Review of 2nd order Circuit

$$\frac{d^2x}{dt^2} + 2\alpha \frac{dx}{dt} + \omega_o^2 x = f(t) \quad \Rightarrow \quad x = X_n + X_f$$

* need two initial conditions $x(t=t_o)$ & $\frac{dx}{dt}(x=x_o)$

Homogeneous equation: $\frac{d^2x}{dt^2} + 2\alpha \frac{dx}{dt} + \omega_o^2 x = 0$

Characteristic equation: $s^2 + 2\alpha s + \omega_o^2 s = 0$

$$\begin{cases} s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} \\ s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2} \end{cases}$$

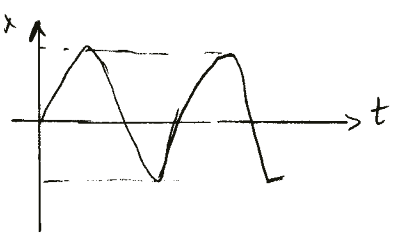
ω_o : undamped natural frequency

$\omega_d = \sqrt{\omega_o^2 - \alpha^2}$ damped natural frequency

$\frac{\alpha}{\omega_o}$: damping ratio.

Undamped (harmonic Oscillator)

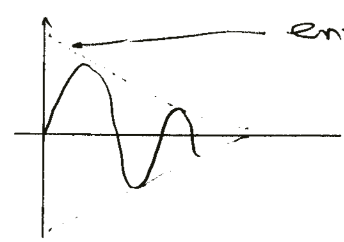
$\alpha = 0$



Underdamped

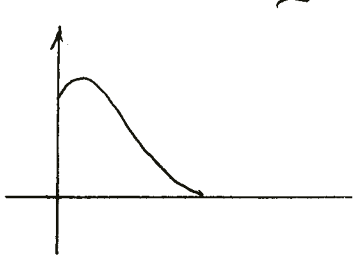
$0 < \alpha < \omega_o^2$

envelope is e



Critically damped ($\alpha = \omega_o$)

* Fastest to steady state



Overdamped $\omega_o^2 < \alpha$