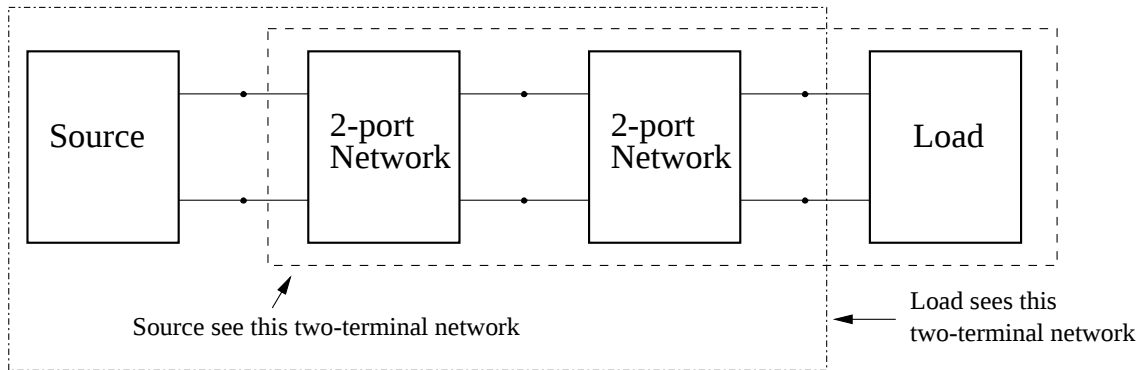
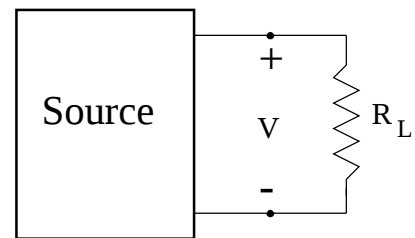


Subcircuit Interfaces and Maximum Power Transfer

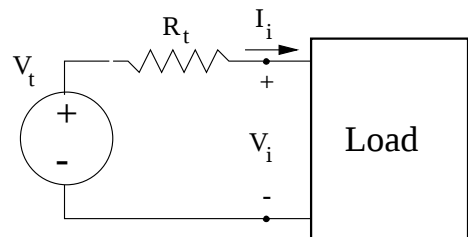
Large electrical and electronic circuits are usually divided into smaller sub-circuits to simplify design and analysis. The strategy of dividing a circuit into individual components works because of the Thevenin Theorem.



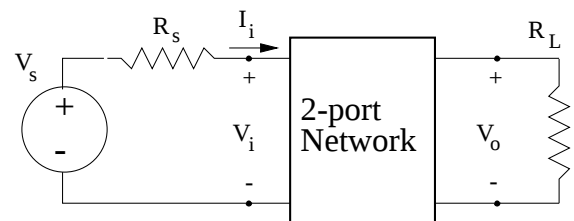
What Source sees: The source sees a two-terminal network. This two-terminal network does not contain an independent source. So it can be reduced to a single impedance.



What Load sees: The load sees a two-terminal network. This two-terminal network contains an independent source. So it can be reduced to its Thevenin equivalent.

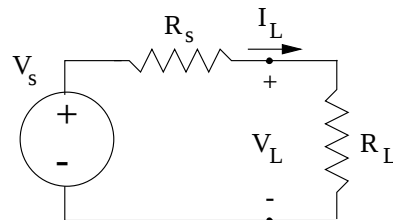


What each two-port network sees: Following the logic above, it's obvious that each two-port network sees a two-terminal network containing an independent source on the input side (can be reduced to a Thevenin form) and a two-terminal network that does not contain an independent source on the output side (so it can be reduced to a single resistor).



Subcircuit Interfaces & Maximum Power Transfer

An important part of strategy of dividing a circuit into individual components is understanding of the interaction and interface between the subcircuits.



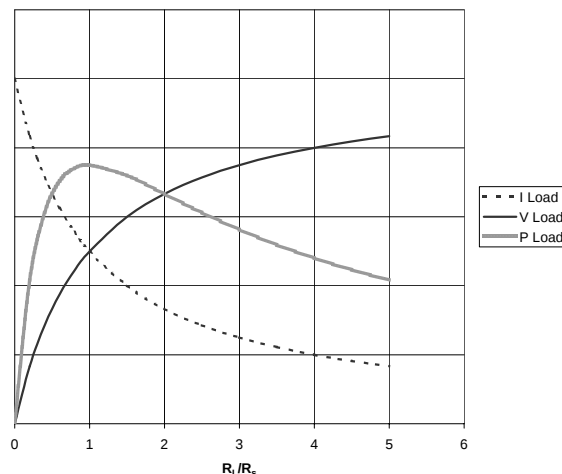
Following the above discussion, one notes that the interface between different subcircuits can be reduced to the simple circuit shown (For example, take the circuit as seen by the load in the previous page and replace load with its equivalent, a resistor). Then,

$$i_L = \frac{v_s}{R_s + R_L}$$

$$v_L = R_L i_L = \frac{R_L}{R_s + R_L} v_s$$

$$P_L = v_L i_L = \frac{R_L}{(R_s + R_L)^2} v_s^2$$

Values of i_L , v_L , and P_L are plotted in the figure. We can see that the load current is maximum when $R_L = 0$ (or effectively, $R_L/R_s \ll 1$) and the voltage on the load is maximum when $R_L \rightarrow \infty$ (or effectively, $R_L/R_s \gg 1$).



In some cases, we are interested in transferring maximum power from a given source (R_s and v_s are known) to a load (*e.g.*, an amplifier driving a speaker). We see from the above the equations that the power transferred to the load is in fact zero when the load current is maximum ($R_L = 0$ leading to $v_L = 0$) or when the voltage on the load is maximum ($R_L \rightarrow \infty$ leading to $i_L = 0$). Maximum power transfer occurs somewhere in between as can be seen from the figure. To find the value of R_L which results in maximum power transfer (v_s and R_s are known), we find derivative of P_L with respect to R_L and set it equal to zero.

$$\frac{dP_L}{dR_L} = \frac{R_s - R_L}{(R_s + R_L)^3} v_s^2$$

$$\frac{dP_L}{dR_L} = 0 \quad \rightarrow \quad R_L = R_s$$

So the power transfer to the load is maximum when $R_L = R_s$, and the maximum transferred power is

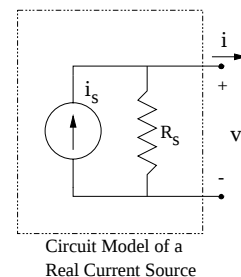
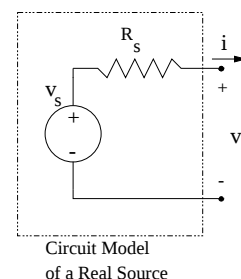
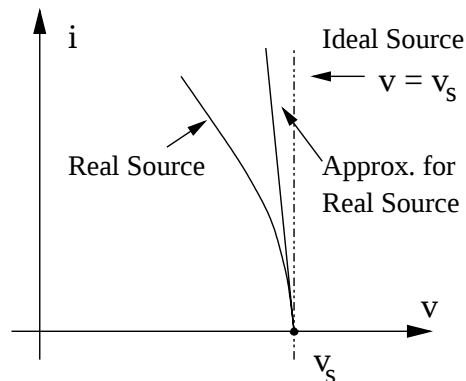
$$P_L|_{Max} = \frac{v_s^2}{4R_L}$$

Real Sources

In an ideal voltage source, the voltage is constant no matter what current is drawn from the source. In a real, practical voltage source (like a battery), however, the output voltage typically decreases as more and more current is drawn, as is shown in the figure. Typically a real source is “rated” for currents below a current i which corresponds to a voltage $v \geq 95\%v_s$ (region near v_s in the figure). For this region, it is a good approximation to model the i - v characteristics of a real source with a straight line. The equation of this line is (using active sign convention):

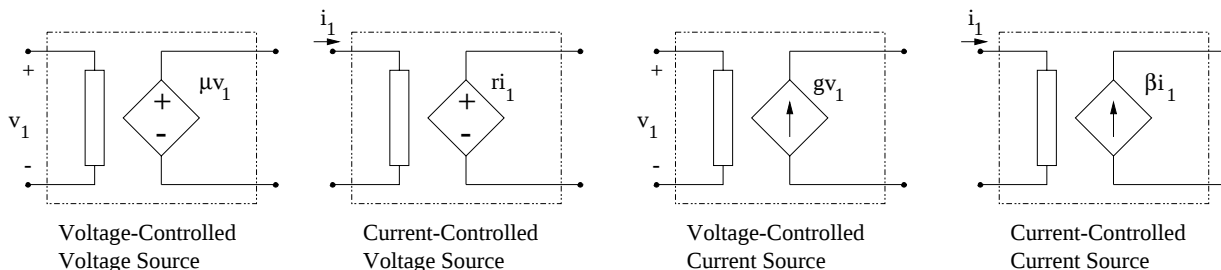
$$v = v_s - R_s i$$

The above approximate i - v characteristics of a real source is a Thevenin form and, therefore, a real source can be modeled with an ideal voltage source, v_s , and a resistance R_s . R_s is called the internal resistance of the source (it is not a real resistor inside the real source!) and is typically small (an ideal voltage source has $R_s = 0$). The same arguments can be applied to “real” current sources. An approximate model for a real current source is in Norton form. R_s is again the internal resistance of the source (and again, it is not a real resistor inside the real source!). For a “real” current source, R_s is typically large (an ideal current source has $R_s \rightarrow \infty$).



Dependent or Controlled Sources

Most analog electronic devices include amplifiers. These are four-terminal devices (two input and two output terminals). The voltage or current in the output terminals are proportional to voltage or current of the input terminals. We need a new circuit element in order to model amplifiers. These elements are “controlled” or “dependent” sources. There are four type of “controlled” sources

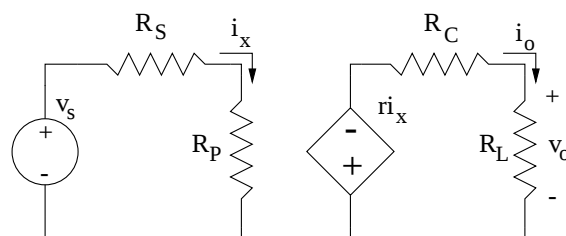


Note that the element located in the input with the controlling current or voltage can be any element: a short circuit, an open circuit, or a resistor.

When one encounters a circuit containing a controlled source, the first step is always to find the “controlling” voltage and current (v_1 or i_1 in the above figures). In some circuits, the control voltage or current is not located near the controlled source in order to simplify circuit drawing. This does not mean that the controlling element is separate from the controlled source. It is essential to always remember that controlled sources are four terminal elements. This means, for example, that you cannot have a subcircuit which include the controlled source but not its controlling element!

Controlled sources behave similar to ideal (or independent) sources. For example, in the voltage-controlled voltage source in the above figure, the output voltage is μv_1 no matter what current is drawn from the circuit. All analysis method developed so far (KVL and KCL, node-voltage and mesh current methods, superposition, *etc.*) can be used for circuits containing controlled sources, and by treating the controlled source similar to an ideal source. In node-voltage and mesh current methods, we need to write an “auxiliary equation” which relates the controlling parameter to node-voltage or mesh current methods as is seen in the examples below.

Example: Find v_o using KVL and KCL:



$$\text{KVL} \quad R_S i_x + R_P i_x - v_s = 0 \quad \rightarrow \quad i_x = \frac{v_s}{R_S + R_P}$$

$$\text{KVL} \quad R_C i_o + R_L i_o + r i_x = 0 \quad \rightarrow \quad i_o = - \frac{r i_x}{R_C + R_L}$$

Substituting for i_x from first equation in the second and noting $v_o = R_L i_o$, we get:

$$v_o = R_L i_o = R_L \left[- \frac{r}{R_C + R_L} \times \frac{v_s}{R_S + R_P} \right]$$

$$v_o = \left[\frac{-r R_L}{(R_S + R_P)(R_C + R_L)} \right] v_s$$

Example: Find v_1 using node-voltage and mesh-current methods.

Node-voltage method:

Circuit has 5 nodes and two voltage sources (one independent and one controlled voltage source). Thus, the number of node-voltage equations is $N_{NV} = 5 - 1 - 2 = 2$. Following our procedure for node-voltage method, we choose the reference node to be the one with most voltage sources attached to it. Then, we can write down the voltage at two of our nodes which are connected to voltage sources:

$$v_A = 16 \quad v_C = 8i_1$$

We then proceed with writing KCL at the other two nodes:

$$\text{Node } v_B: \quad \frac{v_B - 8i_1}{2} + \frac{v_B - v_D}{4} - 1.25 = 0 \quad \rightarrow \quad 3v_B - v_D = 5 + 16i_1$$

$$\text{Node } v_D: \quad \frac{v_D - 0}{4} + \frac{v_D - 16}{2} + \frac{v_D - v_B}{4} - 0.75v_1 = 0 \quad \rightarrow \quad -v_B + 4v_D = 32 + 3v_1$$

Two above equations are two equations in two unknowns (v_B and v_D). But they also contain the control parameters i_1 and v_1 . We need to write two “auxiliary equations” relating these control parameters to our node voltages:

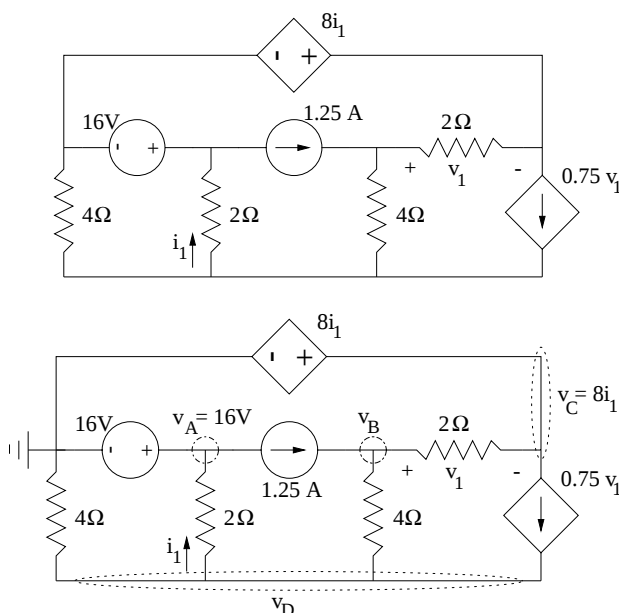
$$i_1 = \frac{v_D - 16}{2} \quad v_1 = v_B - 8i_1 \quad \rightarrow \quad v_1 = +v_B - 4v_D + 64$$

We now substitute for control parameters i_1 and v_1 in our node-voltage equations to get:

$$3v_B - v_D = 5 + 8(v_D - 16) \quad \rightarrow \quad 3v_B - 9v_D = -123$$

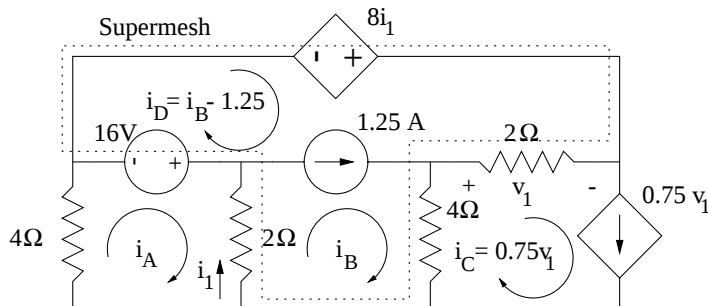
$$-v_B + 4v_D = 32 + 3(+v_B - 4v_D + 64) \quad \rightarrow \quad -4v_B + 16v_D = 224$$

Which can be solved to find the node voltages $v_B = 4$ V and $v_D = 15$ V. The control parameters are: $i_1 = -0.5$ A and $v_1 = 8$ V



Mesh-current method:

The circuit has four meshes and two current sources. So the number of mesh equations is $N_{MC} = 4 - 2 = 2$. Following our procedure for mesh-current method, we can write down two mesh currents using the current sources:



$$i_C = 0.75v_1 \quad i_B - i_D = 1.25 \quad \rightarrow \quad i_D = i_B - 1.25$$

We then proceed with writing KVL at the meshes (one mesh and one supermesh):

$$\text{Mesh } i_A: \quad +4i_A - 16 + 2(i_A - i_B) = 0 \quad \rightarrow \quad 6i_A - 2i_B = 16$$

$$\begin{aligned} \text{Supermesh } i_B \text{ and } i_D: \quad & -8i_1 + 2(i_D - i_C) + 4(i_B - i_C) + 2(i_B - i_A) + 16 = 0 \\ & -8i_1 + 2(i_B - 1.25 - 0.75v_1) + 4(i_B - 0.75v_1) \\ & \quad + 2(i_B - i_A) + 16 = 0 \\ & -2i_A + 8i_B = -13.5 + 8i_1 + 4.5v_1 \end{aligned}$$

Two above equations are two equations in two unknowns (i_A and i_B). But they also contain the control parameters i_1 and v_1 . We need to write two “auxiliary equations” relating these control parameters to our node voltages:

$$i_1 = i_B - i_A$$

$$v_1 = 2(i_C - i_D) = 2(0.75v_1 - i_B + 1.25) \quad \rightarrow \quad v_1 = 4i_B - 5$$

We now substitute for control parameters i_1 and v_1 in our mesh-current equations to get:

$$3i_A - i_B = 8$$

$$-2i_A + 8i_B = -13.5 + 8(i_B - i_A) + 4.5(4i_B - 5) \quad \rightarrow \quad i_A - 3i_B = -6$$

Which can be solved to find the mesh currents: $i_A = 3.75$ A and $i_B = 3.25$ A. The control parameters are: $i_1 = -0.5$ A and $v_1 = 8$ V

Thevenin Equivalent of Subcircuits with Controlled Sources

Two-terminal subcircuits containing controlled sources reduce to Thevenin form. However, care should be taken in doing so. We discussed three methods to find equivalent of a subcircuit. Our first method, source transformation and circuit reduction, does not work with controlled sources. The second method, directly find i - v characteristics of the subcircuit works but is cumbersome (we may have to use for some subcircuits with controlled sources). The third method was to find two of three parameters: R_T (by killing independent sources), $v_t = v_{oc}$ and $i_N = i_{sc}$. Most of the times, the best choice for subcircuits containing controlled sources is to find $v_t = v_{oc}$ and $i_N = i_{sc}$ as described in the example below.

Example: Find the Thevenin equivalent of this subcircuit.

Since the circuit has a controlled source, it is preferred to calculate v_{oc} and i_{sc} .

Finding v_{oc}

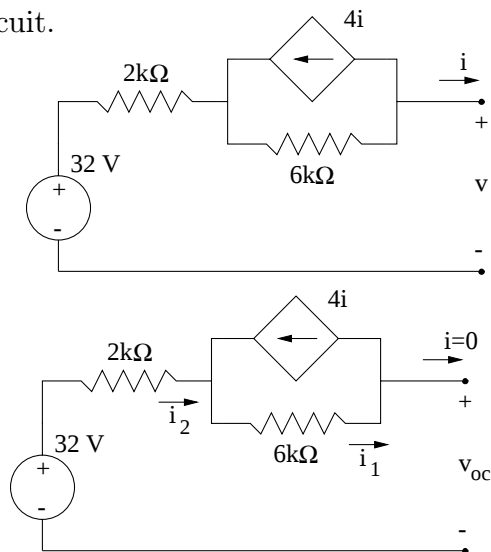
Since the circuit is simple, we proceed to solve it with KVL and KCL (noting $i = 0$):

$$\text{KCL: } -i_1 + i + 4i = 0 \quad \rightarrow \quad i_1 = 0$$

$$\text{KCL: } -i_2 - 4i + i_1 = 0 \quad \rightarrow \quad i_2 = 0$$

$$\text{KVL: } -32 + 2 \times 10^3 i_2 + 6 \times 10^3 i_1 + v_{oc} = 0$$

$$v_T = v_{oc} = 32 \text{ V}$$



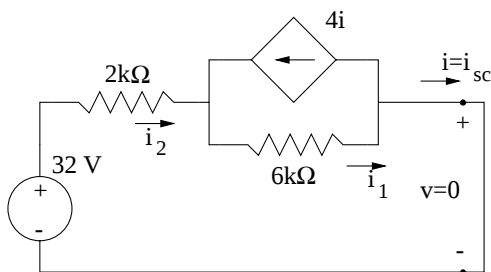
Finding i_{sc}

Since the circuit is simple, we proceed to solve it with KVL and KCL:

$$\text{KCL: } -i_1 + i + 4i = 0 \quad \rightarrow \quad i_1 = 5i_{sc}$$

$$\text{KCL: } -i_2 - 4i + i_1 = 0 \quad \rightarrow \quad i_2 = i_{sc}$$

$$\text{KVL: } -32 + 2 \times 10^3 i_{sc} + 6 \times 10^3 i_{sc} = 0 \quad \rightarrow \quad i_N = i_{sc} = 4 \times 10^{-3} \text{ A} = 4 \text{ mA}$$



Therefore, $v_T = 32 \text{ V}$, $i_N = 4 \text{ mA}$, and $R_T = v_T / i_N = 8 \text{ k}\Omega$.

While finding v_{oc} and i_{sc} is preferred method for most circuits, in some cases, the Thevenin equivalent of the subcircuit is only a resistor (you will find $v_{oc} = 0$ and $i_{sc} = 0$), or only a voltage source (you will find $v_{oc} \neq 0$ but finding i_{sc} leads to inconsistent or illegal circuits), or only a current source (you will find $i_{sc} \neq 0$ but finding v_{oc} leads to inconsistent or illegal circuits). For these cases, one has to either find R_T directly and/or directly find i - v characteristics of the subcircuits as is shown below for the circuit of previous example.

Finding R_T

To find R_T , we kill all independent sources in the circuit. The resulting circuit cannot be reduced to a simple resistor by series/parallel formulas. This is why finding v_{oc} and i_{sc} is the preferred choices for subcircuits containing controlled sources. We can find R_T by attaching an ideal voltage source with a known voltage of v and calculate i . Since the subcircuit should be reduced to a resistor (R_T), we should get $i = -v/(constant)$ where the *constant* is R_T . (Negative sign comes from active sign convention used for Thevenin subcircuit).

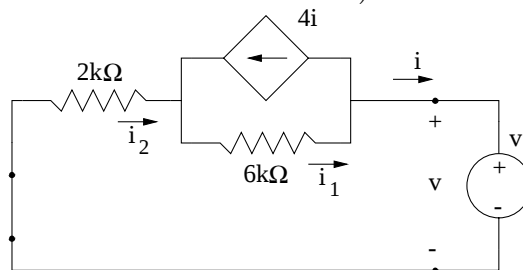
Since the circuit is simple, we proceed to solve it with KVL and KCL:

$$\text{KCL:} \quad -i_1 + i + 4i = 0 \quad \rightarrow \quad i_1 = 5i$$

$$\text{KCL:} \quad -i_2 - 4i + i_1 = 0 \quad \rightarrow \quad i_2 = i$$

$$\text{KVL:} \quad 0 + 2 \times 10^3 i + 6 \times 10^3 i + v = 0$$

$$i = -\frac{v}{8 \times 10^3}$$



Therefore, $R_T = 8 \times 10^3 \Omega = 8 \text{ k}\Omega$.

Note that we could have attached an ideal “current” source with strength of i to the problem, proceeded to calculate v , and would have got $v = -8 \times 10^3 i$.

Finding i - v Characteristics Equation:

As mentioned above, in some cases, we have to directly find the i - v characteristics equation in order to find the Thevenin equivalent of a subcircuit. The procedure is similar to finding R_T . Attach an ideal voltage source to the circuit. Assume v is known and proceed to calculate i in terms of v . Alternatively, one can attach an ideal current source, assume i is known and find v in terms of i . The final expression should look like $v = v_T - iR_T$ and v_T and R_T can be read directly:

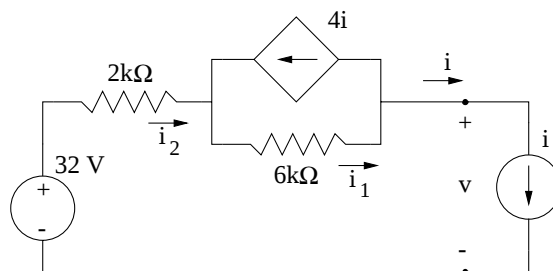
Since the circuit is simple, we proceed to solve it with KVL and KCL:

$$\text{KCL:} \quad -i_1 + i + 4i = 0 \quad \rightarrow \quad i_1 = 5i$$

$$\text{KCL:} \quad -i_2 - 4i + i_1 = 0 \quad \rightarrow \quad i_2 = i$$

$$\text{KVL:} \quad -32 + 2 \times 10^3 i + 6 \times 10^3 i + v = 0$$

$$v = 32 - 8 \times 10^3 i$$



which is the characteristics equation for the subcircuit and leads to $v_T = 32 \text{ V}$, $R_T = 8 \times 10^3 \Omega = 8 \text{ k}\Omega$, and $i_N = v_T/R_T = 4 \text{ mA}$.