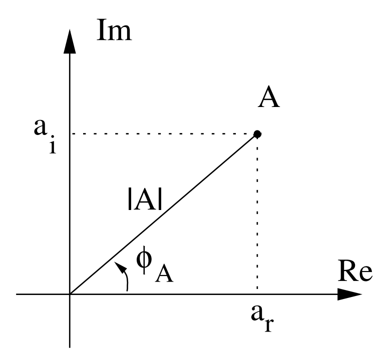
Algebra of Complex Variables

Complex variable A is made of two real numbers:

$$A = a_r + ja_i \qquad a_r \text{ and } a_i \text{ are both real and } j = \sqrt{-1}$$

Since a complex number A is constructed of two real numbers (a_r and a_i), it can be viewed as a point in a two-dimensional plane, called the "complex" plane as is shown. a_r and a_i denote the Cartesian coordinates of the point. In a 2-D plane, points can also be represented by polar coordinates (r and ϕ) or $|A|$ and ϕ_A for the complex number A.

Magnitude of A:	$\|A\|$
Phase of A:	ϕ_A
Real part of A:	$a_r = Re\{A\}$
Imaginary part of A	$a_i = Im\{A\}$
Rectangular form of A	$A = a_r + ja_i$
Polar form of A	$A = \|A\|\angle^{\phi}$



From the diagram, we get:

$$\begin{cases} |A| = \sqrt{a_r^2 + a_i^2} \\ \phi_A = \tan^{-1}\left(\dfrac{a_i}{a_r}\right) \end{cases} \qquad \text{or} \qquad \begin{cases} a_r = |A|\cos(\phi_A) \\ a_i = |A|\sin(\phi_A) \end{cases}$$

Note that ϕ_A angle ranges from -180° to 180° (or from zero to 360°) while $\tan^{-1}$ values range from -90° to 90°. Value of ϕ_A depends on the signs of both a_r and a_i and not the sign of a_i/a_r only. For example, if $a_r = a_i = +1$ as well as $a_r = a_i = -1$, $\tan^{-1}(a_i/a_r) = 45°$, while the correct values are $\phi_A = 45°$ for the first case and $\phi_A = -135°$ for the second case. The following table helps find the correct value of ϕ_A (It assumes that the calculator gives $\tan^{-1}$ values from $-90°$ to $90°$.)

$a_r > 0$	$a_i > 0$	$\rightarrow$	$0° < \phi_A < 90°$	$\tan^{-1}$ value is correct
$a_r > 0$	$a_i < 0$	$\rightarrow$	$-90° < \phi_A < 0°$	$\tan^{-1}$ value is correct
$a_r < 0$	$a_i > 0$	$\rightarrow$	$90° < \phi_A < 180°$	Add 180° to $\tan^{-1}$ value
$a_r < 0$	$a_i < 0$	$\rightarrow$	$-180° < \phi_A < -90°$	Add $-180°$ to $\tan^{-1}$ value

Euler's Formula relates the rectangular and polar form of complex numbers:

Euler's Formula: $e^{j\phi} = \cos\phi + j\sin\phi$

$$A = a_r + ja_i = |A|\cos\phi + j|A|\sin\phi = |A|e^{j\phi} = |A|\angle^{\phi}$$

Algebra of Complex Variables:

Consider two complex variables, $A = a_r + ja_i = |A|\angle^{\phi_A}$ and $B = b_r + jb_i = |B|\angle^{\phi_B}$. Then,

$$A + B = (a_r + ja_i) + (b_r + jb_i) = (a_r + b_r) + j(a_i + b_i)$$

Note: $|A + B| \neq |A| + |B| \qquad \angle^{\phi_{A+B}} \neq \angle^{\phi_A} + \angle^{\phi_B}$

$$A \times B = (a_r + ja_i) \times (b_r + jb_i) = (a_r b_r - a_i b_i) + j(a_r b_i + a_i b_r)$$

$$A \times B = |A|e^{j\phi_A} \times |B| \times e^{j\phi_B} = |A||B|e^{j(\phi_A + \phi_B)}$$

Note: $|A.B| = |A|.|B| \qquad \angle^{\phi_{A.B}} = \angle^{\phi_A} + \angle^{\phi_B}$

$$\frac{A}{B} = \frac{|A|\, e^{j\phi_A}}{|B|\, e^{j\phi_B}} = \frac{|A|}{|B|}\, e^{j(\phi_A - \phi_B)}$$

Note: $\left|\frac{A}{B}\right| = \frac{|A|}{|B|} \qquad \angle^{\phi_{A/B}} = \angle^{\phi_A} - \angle^{\phi_B}$

Complex Conjugate:

$$A = a_r + ja_i = |A|\angle^{\phi_A}$$

$$A^* = a_r - ja_i = |A|\angle^{-\phi_A} \qquad \text{Complex Conjugate of } A$$

$$A + A^* = (a_r + ja_i) + (a_r - ja_i) = 2a_r$$

$$A - A^* = (a_r + ja_i) - (a_r - ja_i) = j2a_i$$

$$A.A^* = (|A|\angle^{\phi_A}) \times (|A|\angle^{-\phi_A}) = |A|^2 \qquad \text{A real number}$$

To find the ratio of two complex number, we multiply the both nominator and denominator with the complex conjugate of the denominator:

$$\frac{A}{B} = \frac{A.B^*}{B.B^*} = \frac{1}{|B|^2}(A.B^*)$$

$$\frac{A}{B} = \frac{a_r + ja_i}{b_r + jb_i} = \frac{(a_r + ja_i)(b_r - jb_i)}{(b_r + jb_i)(b_r - jb_i)} = \frac{(a_r b_r + a_i b_i) + j(-a_r b_i + a_i b_r)}{b_r^2 + b_i^2}$$