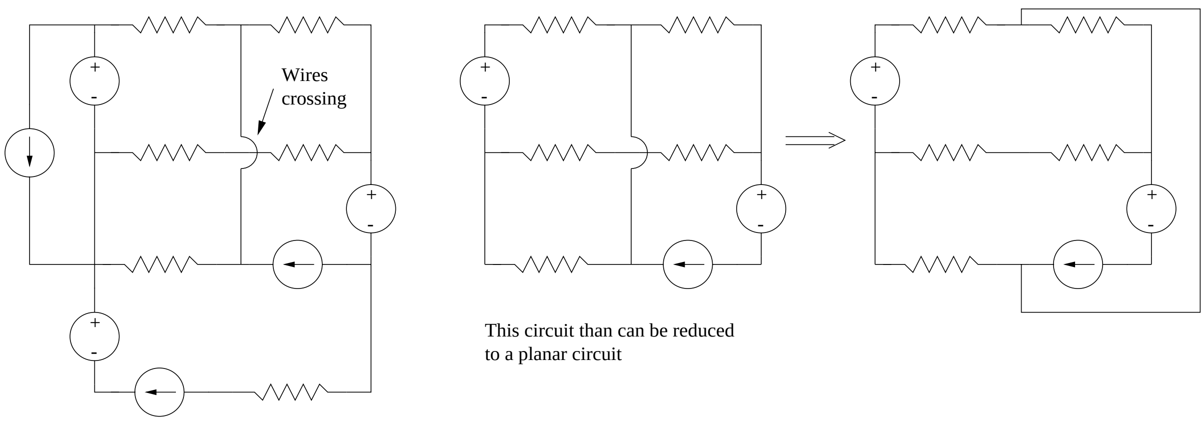
Mesh-Current Method

The mesh-current is analog of the node-voltage method. We solve for a new set of variables, mesh currents, that automatically satisfy KCLs. As such, mesh-current method reduces circuit solution to writing a bunch of KVLs.

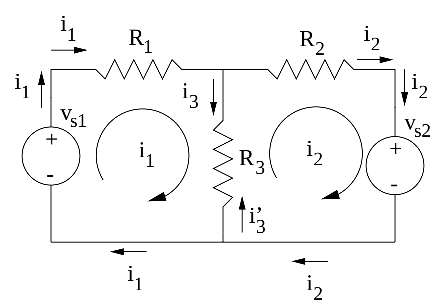
Note: Mesh-current method only works for planar circuits: circuits that can be drawn on a plane (like on a paper) without any elements or connecting wires crossing each other as shown below. Note that in some cases a circuit that looks non-planar can be made into a planar circuit by moving some of the connecting wires (see figure)



A Non-planar Circuit

This circuit than can be reduced to a planar circuit

Mesh-current method is best explained in the context of example circuit below.



- A **mesh** is defined as a closed path (a loop) that contains no closed path within it.
- **Mesh current** is the current that circulates in the mesh *i.e.,*

a) if an element is located on a single mesh (such as R_1, R_2, v_{s1}, and v_{s2}) it carries the same current as the mesh current,

b) If an element is located on the boundary of two meshes (such as R_3), it will carry a current that is the algebraic sum of the the two mesh currents:

$$\begin{aligned} i_3 &= i_1 - 2_2 \\ i_3' &= i_2 - i_1 \end{aligned}$$

In this way, KCLs are automatically satisfied. In addition, as we can write current in each element in terms of mesh currents, we can use *i*-*v* characteristics of element to write the voltage across each element in terms on mesh currents. Therefore, we need only to write KVLs in terms of mesh currents.

In the circuit above, KVLs give:

Mesh 1: $R_1 i_1 + R_3(i_1 - i_2) - v_{s1} = 0 \quad \rightarrow \quad (R_1 + R_3)i_1 - R_3 i_2 = v_{s1}$

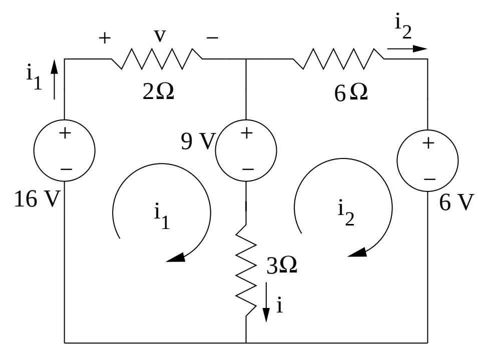
Mesh 2: $R_3(i_2 - i_1) + R_2 i_2 + v_{s1} = 0 \quad \rightarrow \quad -R_3 i_1 + (R_2 + R_3)i_2 = -v_{s2}$

or in matrix form,

$$\begin{bmatrix} R_1 + R_3 & -R_3 \\ -R_3 & R_2 + R_3 \end{bmatrix} \bullet \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} v_{s1} \\ -v_{s2} \end{bmatrix} \quad \rightarrow \quad \mathbf{R} \bullet \mathbf{i} = \mathbf{v_s}$$

Which is similar in form to matrix equation found for node-voltage method. **i** is the array of mesh currents (unknowns), $\mathbf{v_s}$ is the array of independent voltage sources, and **R** is the resistance matrix and is symmetric. The diagonal element, R_{jj}, is the sum of resistance around mesh j and the off-diagonal elements, R_{jk}, are the sum of resistance shared by meshes j and k.

Example Find i and v.
Using mesh-current method:



Mesh 1: $2i_1 + 9 + 3(i_1 - i_2) - 16 = 0$

Mesh 2: $6i_2 + 6 + 3(i_2 - i_1) - 9 = 0$

$$\begin{cases} 5i_1 - 3i_2 = 7 \\ -3i_1 + 9i_2 = 3 \end{cases} \quad \rightarrow \quad \begin{cases} i_1 = 2 \text{ A} \\ i_2 = 1 \text{ A} \end{cases}$$

The problem unknowns, i and v can now be found from the mesh currents:

$$i = i_1 - i_2 = 1 \text{ A}$$

$$v = 2i_1 = 4 \text{ V}$$

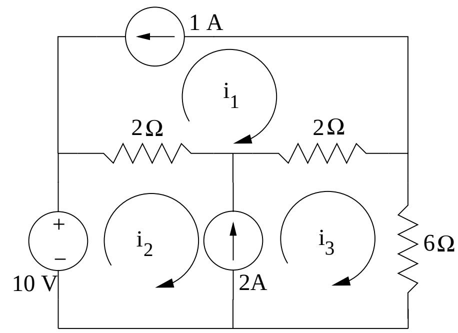
Mesh currents method for circuits with current sources

Because of i-v characteristics of a current source does not specify its voltage, we have to modify mesh-current method. This is best seen in the example below:

From the circuit, we note:

• If a current source is located on only <u>one</u> mesh (1-A ICS in the circuit), the mesh current can be directly found from the current source and we do not need to write any KVL:

$$i_1 = -1 \text{ A}$$



• If a current source is located on the boundary between two meshes (2-A ICS in the circuit), KVL on these meshes (mesh 2 or 3 in the above circuit) contain the voltage across the 2-A ICS which is unknown. We need two equations to substitue for the two KVLs on meshes 2 and 3 that are not useful now. The first one is found from the *i*-*v* characteristics of the current source (its current should be 2 A):

$$i_3 - i_2 = 2 \text{ A}$$

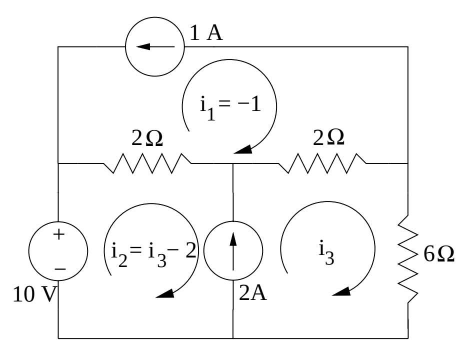
The second equation can be found by noting that KVL can be written over any closed loop. We define a **supermesh** as the combination of two meshes which have a current source on their boundary as shown in the figure. While KVL on mesh 2 or on mesh 3 both include the voltage across the 2-A current source that is unknown, KVL on the supermesh does not include that:

Supermesh 2&3: $\quad 2(i_2 - i_1) + 2(i_3 - i_1) + 6i_3 - 10 = 0 \quad \rightarrow \quad -4i_1 + 2i_2 + 8i_3 = 10$

$$\begin{cases} i_1 &= 1 \\ i_2 - i_3 &= -2 \\ 2i_2 + 8_i3 &= 10 + 4i_1 = 6 \end{cases} \quad \rightarrow \quad \begin{cases} i_1 &= 1 \\ i_2 &= i_3 - 2 \\ 2(i_3 - 2) + 8_i3 &= 10 \end{cases}$$

which results in $i_1 = 1$ A, $i_2 = -1$ A, and $i_3 = 1$ A,

Note that we could have used $i_1 = 1$ and $i_2 = i_3 - 2$ equations directly on the meshes as shown in the figure and wrote only the KVL on the supermesh:



supermesh 2&3: $\quad 2[(i_3 - 2) - (-1)) + 2[i_3 - (-1)] + 6i_3 - 10 = 0$

$\quad i_3 = 1 \text{ A} \quad \rightarrow \quad i_2 = 1 - 2 = -1 \text{ A}$

Recipe for Mesh-Current Method

1. Check if circuit is planar.
2. Identify meshes, mesh currents, and supermeshes.
   a) Rearrange the circuit if possible to position current source on a single mesh.
   b) Use *i*-*v* characteristic equations of ICS to find mesh currents and reduce the number of unknowns.
3. Write KVL at each mesh and supermesh.
4. Solve for mesh currents.
5. Calculate problem unknowns from mesh currents. If you need to calculate the voltage across a current source you may have to write KVL around a mesh containing the current source.
6. For consistency and elimination of errors, always mark all mesh currents in clockwise direction and write down KVLs in the same direction.

Comparison of Node-voltage and Mesh-current methods

Node-voltage and mesh-current are powerful methods that simplify circuit analysis substantially. They are methods of choice in almost all cases (except for very simple circuits or special circuits). Examination of the circuit can also tell us which of the two methods are best suited for the circuit at hand. We always want to reduce the circuit equations into the smallest number of equations in smallest number of unknowns. The number of equations from node-voltage method, N_{NV} and mesh current method, N_{MC} are given by:

$$N_{NV} = N_{node} - 1 - N_{VS}$$
$$N_{MC} = N_{mesh} - N_{CS}$$

where N_{VS} and N_{CS} are numbers of voltage and current sources, respectively. Thus, always inspect the circuit, find N_{VS} and N_{CS}, and proceed with the method that results in the smallest number of equations to solve.

Note: You need to check to ensure that the circuit is a planar circuit. If it is not one cannot use mesh-current method and should use node-voltage method.

Additive properties and Superposition

In solving any linear circuit, we always end up with a set of simultaneous linear equations of the form

$$\mathbf{A} \bullet \mathbf{x} = \mathbf{s}$$

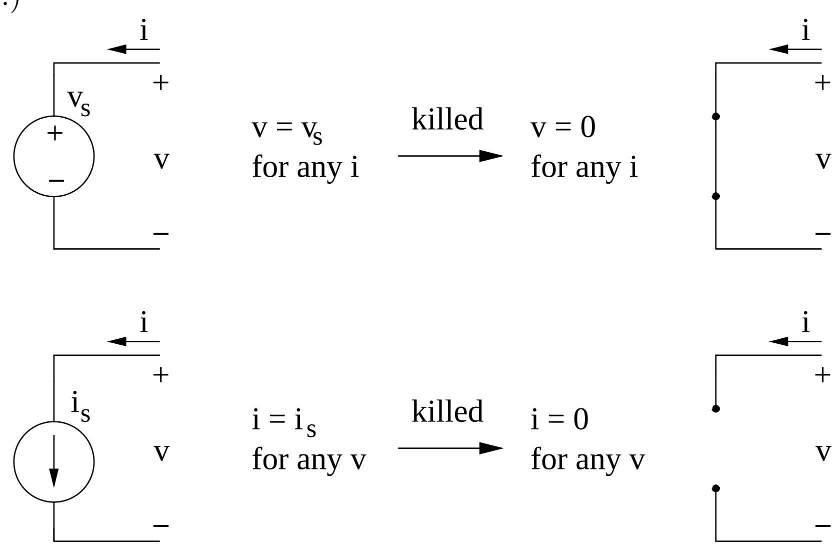
$\mathbf{A}$:	matrix of resistances or conductances
$\mathbf{x}$:	array of circuit variables, i and/or v, (unknowns)
$\mathbf{s}$:	array of independent sources

Linear algebra tells that if we know the solution to $\mathbf{A} \bullet \mathbf{x} = \mathbf{s_1}$ to be $\mathbf{x_1}$ (*i.e.,* $\mathbf{A} \bullet \mathbf{x_1} = \mathbf{s_1}$) and if we know the solution to $\mathbf{A} \bullet \mathbf{x} = \mathbf{s_2}$ to be $\mathbf{x_2}$ (*i.e.,* $\mathbf{A} \bullet \mathbf{x_2} = \mathbf{s_2}$), then the solution to $\mathbf{A} \bullet \mathbf{x} = \mathbf{s_1} + \mathbf{s_2}$ is $\mathbf{x} = \mathbf{x_1} + \mathbf{x_2}$ because:

$$\mathbf{A} \bullet \mathbf{x_1} + \mathbf{A} \bullet \mathbf{x_2} = \mathbf{s_1} + \mathbf{s_2}$$

$$\mathbf{A} \bullet (\mathbf{x_1} + \mathbf{x_2}) = \mathbf{s_1} + \mathbf{s_2}$$

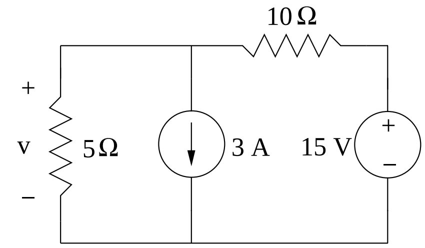
In a linear circuit this property means that:
Additive property of linear circuit or Principle of Superposition: If a linear circuit is driven by more than one independent source, the response of the circuit can be written as the sum of the responses of the circuit to individual sources with all other sources "killed" (*i.e.,* their strength set to zero.)



Note that "killing" a source does not mean "removeing" it.

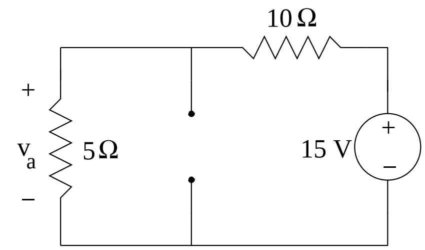
Example: Find v by superposition.

Because we have two independent sources, we first "kill" the current source to arrive at circuit "a" and then we kill the voltage source to arrive at circuit "b". By superposition, $v = v_a + v_b$



Circuit "a" is a voltage divider circuit and v_a can be written down directly as

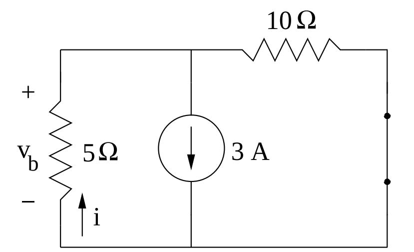
$$v_a = \frac{5}{5+10} \times 15 = 5 \ V$$



Circuit "b" is a current divider circuit and current i can be written down directly as

$$i = \frac{1/5 + 1/10}{1/5} \times 3 = 2 \ A$$

$$v_b = -5i = -10 \ V$$



Thus, $v = v_a + v_b = 5 - 10 = -5$ V.

Note: Using superposition results in slightly simpler circuits (one element is replaced with either a short or open circuit) but more circuits. In general superposition requires more work than node-voltage or mesh-current methods. Superposition is used:

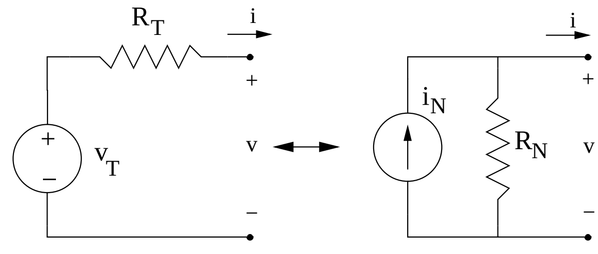
a) If sources are fundamentally different (*e.g.,* dc and ac sources as we see later). In this case superposition may be the only choice,

b) If circuit is repetitive (see example 3-12 text book) such that circuits resulting from applying superposition look identical and, thus, we need only to solve one circuit.

Reduction of two-terminal sub-circuits to Thevenin form

Recall Thevenin and Norton forms and the fact that they are equivalent. The convention is to write the *i*-*v* characteristics of Thevenin/Norton forms with <u>active</u> sign convention:

$$v = v_T - R_T i \qquad i = i_N - \frac{v}{R_N}$$

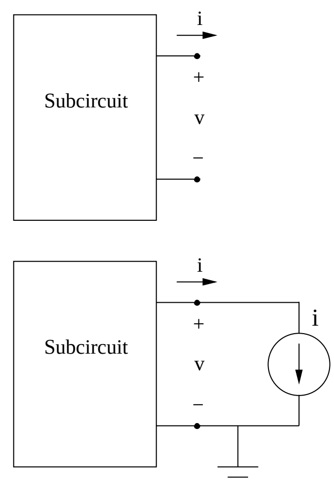


Thevenin Form   Norton Form

Equivalent if $R_T = R_N$ and $v_T = i_N R_T$

We used the equivalency of Norton and Thevenin forms in circuit reduction.

Recall our discussion of equivalent elements and subcircuits. We can replace any two-terminal subcircuit with another one as long as they have the same *i*-*v* characteristics. We will show below that the *i*-*v* characteristics of any two-terminal element containing linear elements is in Thevenin form. First let's examine <u>how to find *i*-*v* characteristics of a two-terminal element</u>.



In order to find the *i*-*v* characteristics of a resistor in the lab, we connect a voltage sources (with adjustable strength) to its terminal, change the strength of the voltage of the source and measure the current flowing through the resistor. After a sufficient number of pairs of i and v are measured, we can plot the result and deduce $v = Ri$. We can perform similar, but mathematical, experiment to find the *i*-*v* characteristics of a two-terminal element. Attach a voltage source across its terminal with a strength v. Solve the circuit and calculate current i which will be in terms of v (*i*-*v* characteristics!). Alternatively, we can attach a current source with strength i to the subcircuit and solve for v in terms i as is shown. This is the general method to "calculate" the *i*-*v*characteristics of a two-terminal element.

Suppose we used node-voltage method and assign the ground as shown. After writing all of node-voltage equations, we will get:

$$\mathbf{G} \bullet \mathbf{v} = \mathbf{i_s}$$

G : Conductance matrix

v : Array of node voltages

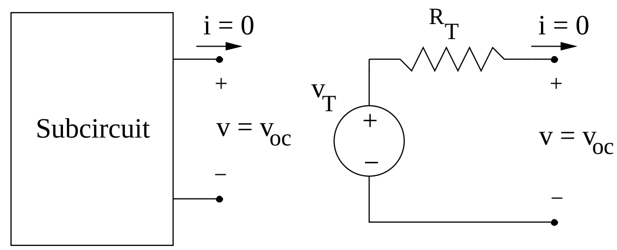
$\mathbf{i_s}$: Array of independent current sources

If we choose as node no. 1 to be the positive terminal of the subcircuit, the node-voltage array will be a column vector, $\mathbf{v} = [v, v_2, v_3, ..]$ with voltage v as its first element. The current source array $\mathbf{i_s}$ also will have i as its first element. Now, if we solve the above matrix equation and denoting the inverse matrix of $\mathbf{G}$ as $\mathbf{G^{-1}}$, we get:

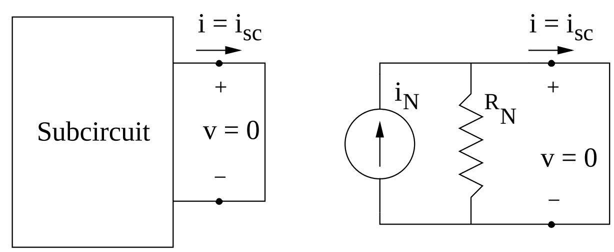
$$\begin{bmatrix} v \\ v_2 \\ ... \\ v_n \end{bmatrix} = \begin{bmatrix} G_{11}^{-1} & G_{12}^{-1} & ... & G_{1n}^{-1} \\ G_{21}^{-1} & G_{22}^{-1} & ... & G_{2n}^{-1} \\ ... & ... & ... & ... \\ G_{n1}^{-1} & G_{n2}^{-1} & ... & G_{nn}^{-1} \end{bmatrix} \bullet \begin{bmatrix} i \\ i_{s2} \\ ... \\ i_{sn} \end{bmatrix}$$

The first row of this matrix equation reduces to $v = -C_1 i + C_2$ where C_1 and C_2 are two constants (Since all G and i_s are constants). C_1 should be a resistance (call it R_T) and C_2 should be a voltage (call it V_T). Thus: **the i-$v$characteristics of any two-terminal element containing linear elements is in the Thevenin form of $v = v_T - R_T i$.**

Next, consider the two-terminal subcircuit and its Thevenin equivalent (they have exactly same i-v characteristics). Let the current $i = 0$ and calculate v, *i.e.,* calculate the voltage across the terminals of the subcircuit while the terminal are open circuit. This voltage is called the open circuit voltage, v_{oc}. Examination of the Thevenin form shows that if $i = 0$, $\underline{v_T = v_{oc}}$.



Next, consider the two-terminal subcircuit and its Norton equivalent (they have exactly same i-v characteristics). Let the voltage $v = 0$ and calculate i, *i.e.,* calculate the current while the subcircuit terminals are shorted. This current is called the short circuit current, i_{sc}. Examination of the Norton form shows that if $v = 0$, $\underline{i_N = i_{sc}}$.



Lastly, examination of the matrix equation above shows that the Thevenin resistance depends of conductance matrix only. Thus, **if one "kills" all of the sources in the subcircuit, the remaining circuit should be equivalent to the Thevenin's resistance**. The above bold/underlined statements constitute the Thevenin's Theorem

Finding Equivalent Thevenin/Norton Forms:

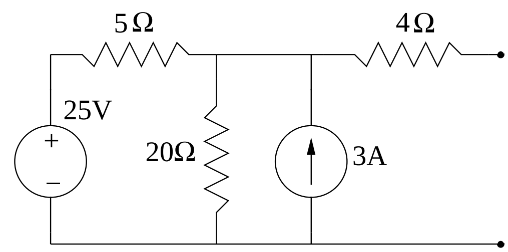
Three methods are available:

Method 1: Use source transformation and circuit reduction to reduce the circuit to a Thevenin/Norton form. This is a cumbersome method, does not always work, and should be used only on simple circuits.

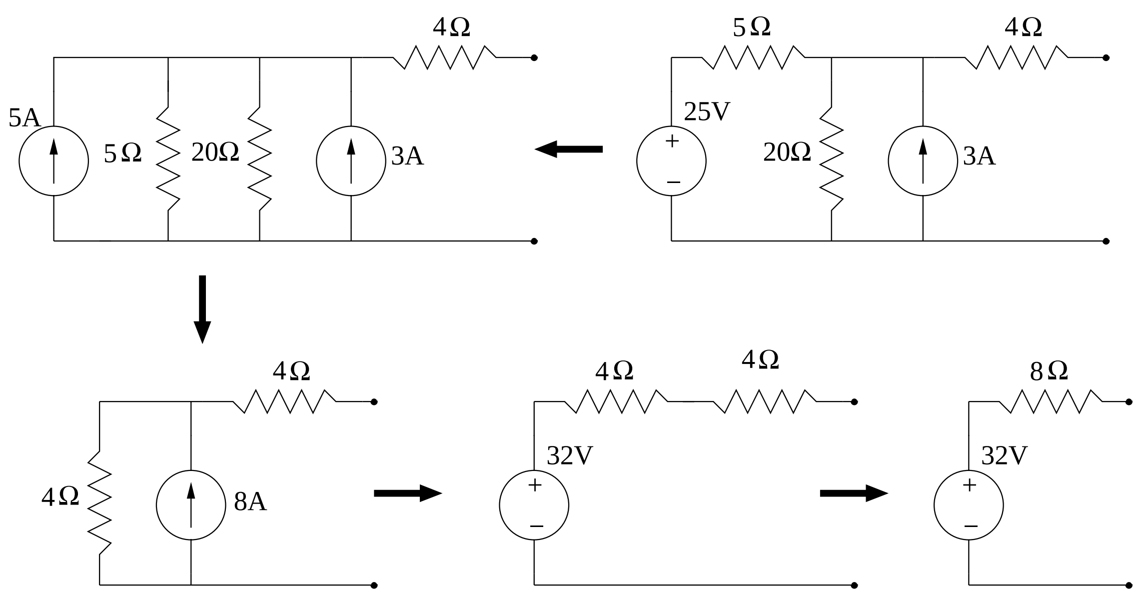
Method 2: Directly find i-v characteristics of the subcircuit by attaching a current source or a voltage source to the circuit as discussed above. This method always work. The drawback is that the circuit has to be solved analytically.

Method 3: Compute <u>two</u> of the following quantities by solving the appropriate circuits: $v_T = v_{oc}$, $i_N = i_{sc}$, and R_T by killing the sources. The third parameter is found from $v_T = R_T i_N$. This is the best method and with a few exceptions, always work.

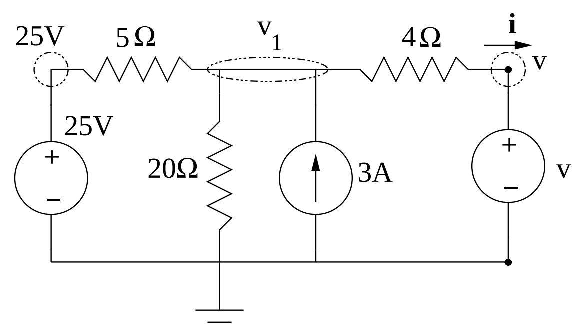
Example: Find Thevenin equivalent of this subcircuit:



Method 1: Source transformation and circuit reduction



Method 2: Directly find i-v characteristics. We attach a voltage source with strength v to the output terminals as shown. Assume v is known and solve for i. Circuit has 4 nodes and two voltage sources, so the number of equation for node-voltage method, $N_{NV} = 4 - 1 - 2 = 1$. Circuit has three meshes and 1 current source, so the number of equations for mesh-current method is $N_{MC} = 3 - 1 = 2$. So, better to do node-voltage method.



$$\frac{v_1 - v}{4} - 3 + \frac{v_1 - 0}{20} + \frac{v_1 - 25}{5} = 0 \qquad \rightarrow \qquad 5v_1 - 5v - 60 + v_1 + 4v_1 - 100 = 0$$

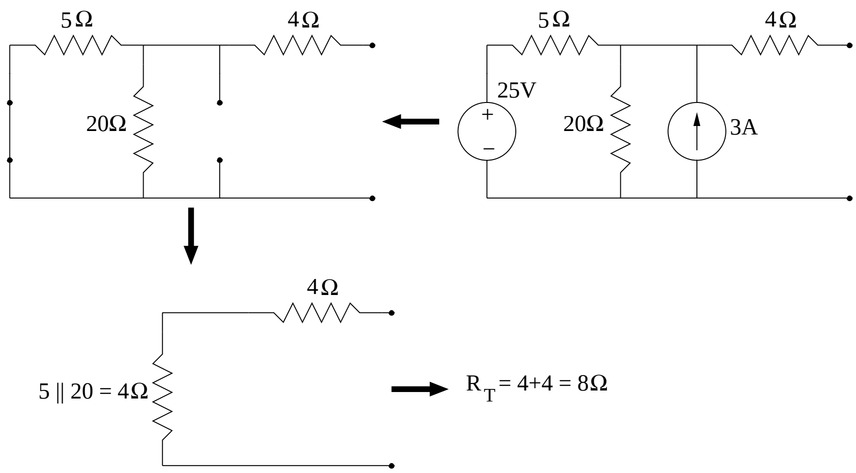
$$v_1 = 16 + 0.5v$$

$$i = \frac{v_1 - v}{4} = \frac{16 + 0.5v - v}{4} \qquad \rightarrow \qquad v = 32 - 8i \qquad (i\text{-}v \text{ characteristics!})$$

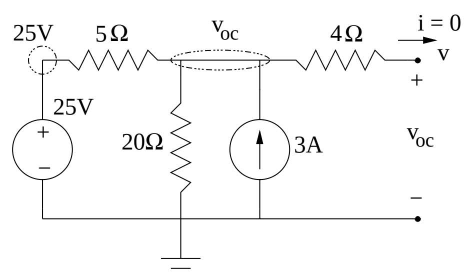
$$v_T = 32 \text{ V} \qquad R_T = 8\ \Omega$$

Method 3: Thevenin's Theorem: find two of the following three: R_T, v_{oc}, and i_{sc}.

a) Find R_T by "killing" the sources



b) Find $v_T = v_{oc}$ (set $i = 0$)
Using node-voltage method (note that the voltage drop across the 4 Ω resistor is zero.

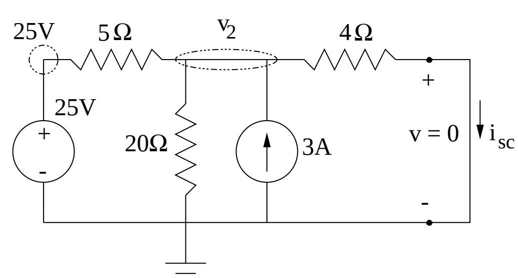


$$\frac{v_{oc} - 25}{5} + \frac{v_{oc} - 0}{20} - 3 = 0$$

$$4v_{oc} - 100 - v_{oc} - 60 = 0$$

$$v_T = v_{oc} = 32 \text{ V}$$

c) Find $i_N = i_{sc}$ (set $v = 0$)
Using node-voltage method:



$$\frac{v_2 - 25}{5} + \frac{v_2 - 0}{20} - 3 + \frac{v_2 - 0}{4} = 0$$

$$4v_2 - 100 + v_2 - 60 + 5v_2 = 0$$

$$v_2 = 16 \text{ V} \qquad \rightarrow \qquad i_N = i_{sc} = \frac{v_2}{4} = 4 \text{ A}$$