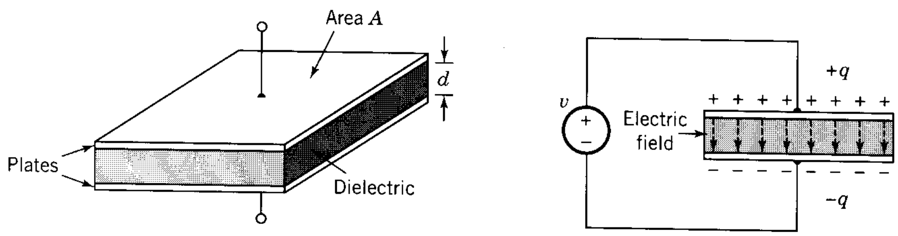
Dynamic or Time-Dependent Circuits

In this section we discuss circuits that include capacitors and inductors. The i-v characteristics of these two elements include derivatives or integral of either i or v resulting in time-dependent circuits.

Capacitor



(a) Essential parts of a capacitor

(b) Displaced charge and electric field

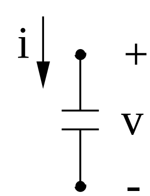
A typical capacitor is made of two parallel conducting plates separated by an insulator. If we connect a capacitor to a voltage source, as is shown above, electrons travel from the voltage source to the capacitor and charge one of the plates with excess electrons. The electric field from these electrons repel electrons from the opposing plate and equal but positive charge develops on that plate. The repelled electrons from the positive plate travel to the source. As such, the total charge stored in a capacitor is zero but consists of two separated amount of equal charges with different polarity. In this manner, a capacitor can store energy in the form of electric field between the two plates, which in circuit theory language, is proportional to the voltage between the two plates.

For most parallel plate capacitors, the voltage across the capacitor is directly proportional to the charge accumulated in the capacitor. The constant of proportionality is the capacitance. Note that the total charge stored in a capacitor is zero, the charge mentioned here is the charge on one of the plates (the plate marked with + of voltage v).

$$C = \frac{q}{v} \qquad \text{Capacitance } C\text{, Unit: Farad (F)}$$

Most commercial capacitors are in μF (10^{-6} F), nF (10^{-9} F), and pF (10^{-12} F) range.

As the number of electrons flowing into the negative plate is equal to the number of electrons flowing out of the positive plate, the current flowing into one plate of capacitor is exactly equal to the current flowing out of the other plate. The i-v characteristics equation for a capacitor (using passive sign convention) is



$$q = Cv \quad \rightarrow \quad i = \frac{dq}{dt} = C\frac{dv}{dt}$$

Note that the current i flows as long as the voltage (and charge, $q = Cv$) changes in time. At the DC steady-state condition, $dv/dt = 0$ and capacitor current, $i = 0$, and capacitor acts as an open circuit.

Energy stored in a capacitor, W, can be found from the definition of electric power:

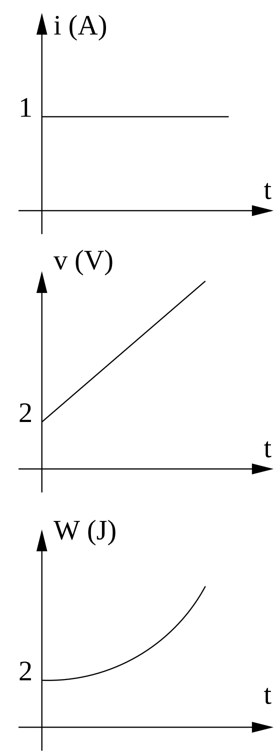
$$\begin{aligned}
\frac{dW}{dt} &= P = vi \\
W(t) - W(t_0) &= \int_{t_0}^{t} P dt' = \int_{t_0}^{t} vidt' = \int_{t_0}^{t} Cv\frac{dv}{dt'}dt' \\
W(t) - W(t_0) &= \frac{1}{2}C[v^2(t) - v^2(t_0)]
\end{aligned}$$

Since at $t_0 \to -\infty$, the voltage across and energy stored in the capacitor are zero, we get:

$$W(t) = \frac{1}{2}Cv^2(t)$$

Note: Unless the voltage waveform across the capacitor is specified, we need an initial condition (voltage at some time t_0) in order to calculate current, voltage, and power of a capacitor. (The condition of $v(t_0 \to -\infty) = 0$ is not usually useful.)

Example: Consider a 1-F capacitor with $v(t = 0) = 2$ V. The current flowing in the capacitor is shown below. Find the voltage across the capacitor and its stored energy as a function of time.



Integrating the i-v characteristics equation of capacitor in time (from t_0 to t) , we get

$$v(t) - v(t_0) = \frac{1}{C}\int_{t_0}^{t} i(t)dt$$

The current waveform is given and is $i(t) = 1$ (for $t > 0$). The initial condition is specified at time $t_0 = 0$ to be $v(t_0) = v(0) = 2$. Thus,

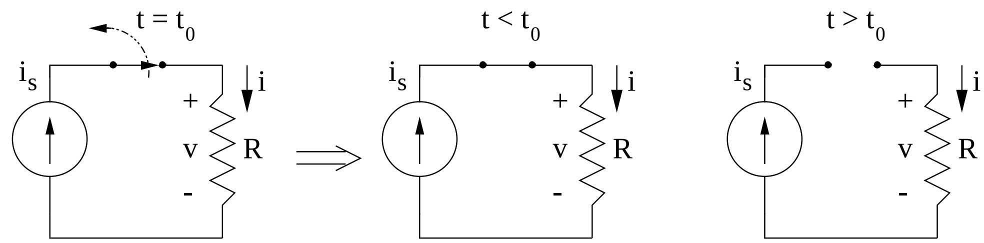
$$v(t) - 2 = \frac{1}{1}\int_{0}^{t} 1dt = t \qquad \to \qquad v(t) = t + 2 \text{ V}$$

The graph of $v(t)$ is shown. The energy stored in the capacitor is:

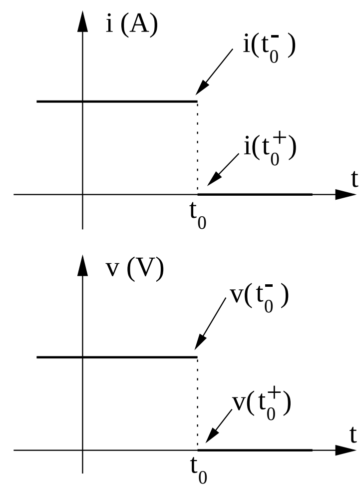
$$W = \frac{1}{2}Cv^2 = 0.5(t+2)^2 \text{ J}$$

Time dependent circuits either have a time-dependent source (we will examine circuit with sinusoidal sources later) and/or have a source which is "switched" on or off at some time. In all switching circuits, we assume that switch is thrown instantaneously. This means that for these switched circuit, the voltage or current waveform will have a discontinuity in time.

For example, consider the circuit shown. The switch is initially closed and is opened at time t_0. For $t < t_0$, the switch is closed. By KCL, $i = i_s$ and a voltage of $v = Ri_s$ appears across the resistor. For $t > t_0$, $i = 0$ and $v = 0$. There is a discontinuity in voltage and current at time $t = t_0$ when the switch is thrown.

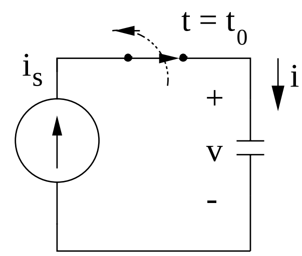


Values of current and voltage at the exact switching time is undefined. The values are known just prior to the throwing the switch and just after that. We denote that time just prior to throwing the switch as t_0^- ($t_0^- = t_0 - \epsilon$ with ϵ being infinitesimally small). We denote the that time just after throwing the switch as t_0^+ ($t_0^+ = t_0 + \epsilon$). In this sense, current and voltage are well defined at t_0^- and t_0^+. In the circuit shown, both current and voltage have a discontinuity at $t = t_0$ as $i(t_0^-) \neq i(t_0^+)$ and $v(t_0^-) \neq v(t_0^+)$.

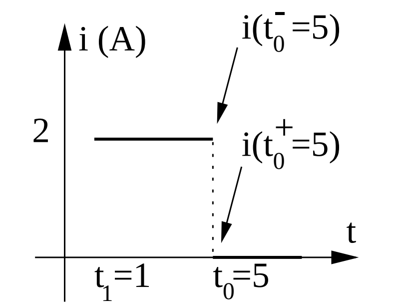


This behavior is in general true for all resistive circuit. There is a discontinuity in both voltage and current waveforms at the time that the switch is thrown. This is not true for circuits with capacitors (or inductors). For capacitors, the *i*-*v* characteristics equation $i = C dv/dt$ implies that if there were a discontinuity in the voltage, an infinite amount of current should flow through the capacitor. Therefore, **the voltage waveform across a capacitor should be continuous.** This means that if a switch is thrown at time t_0, we should have $v(t_0^-) = v(t_0^+)$. Note the capacitor current waveform can have a discontinuity.

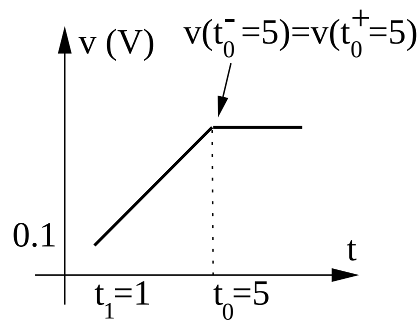
Example: The switch in this circuit is opened at time $t_0 = 5$ s. Find the voltage across the $C = 20$ F capacitor if $i_s = 2$ A and the voltage across the capacitor is $v(t_1 = 1\text{ s}) = 0.1$ V



Since the initial condition is specified at $t = 1$ s, we consider the circuit for $t \geq 1$ s. For $1 < t < t_0 = 5$, the switch is closed and, by KCL, the current in the capacitor is $i = i_s = 2$ A. For $t_0 = 5 < t$, the switch is open and, by KCL, the current in the capacitor is $i = 0$. The resulting current waveform is shown in the figure.



To find the voltage waveform, we integrate the i-v characteristics of the capacitor. We need to consider $1 < t < t_0 = 5$ and $t_0 = 5 < t$ regions separately as the current has different functional form.



For $t_1 = 1 < t < t_0^- = 5$, we integrate the capacitor i-v characteristics equation from time t_1 to t

$$i = C\,\frac{dv}{dt}$$

$$\int_{t_1}^{t} \frac{dv}{dt'} dt' = \frac{1}{C}\,\int_{t_1}^{t} i(t')dt'$$

$$v(t) - v(t_1) = \frac{1}{20}\int_{1}^{t} i(t')dt' = 0.05\int_{1}^{t} 2dt' = 0.05 \times 2t'|_1^t = 0.1(t-1)$$

$$v(t) = 0.1 + 0.1(t-1) = 0.1t$$

So in the time interval $t_1 = 1 < t < t_0 = 5$, voltage across the capacitor starts at $v(t_1 = 1\text{ s}) = 0.1$ V and increases to $v(t_0^- = 5\text{ s}) = 0.5$ V.

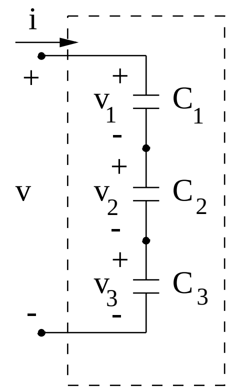
For $t_0^+ = 5 < t$, we integrate the capacitor i-v characteristics equation from time t_0 to t $(i = 0)$, and use no-jump condition: $v(t_0^+) = v(t_0^-) = 0.5$ V.

$$v(t) - v(t_0^+) = \frac{1}{20}\int_{5}^{t} i(t')dt' = 0$$

$$v(t) = v(t_0^+) = 0.5$$

Note that the voltage across the capacitor was continuous while the current waveform included a jump at the switching time.

Capacitors in Series



The equivalent to a subcircuit consisting of several capacitors in series can be found using the same procedure as that used for series/parallel resistors–we need to find the *i*-*v* characteristics of the subcircuit.

KCL: capacitors in series all carry current i

KVL: $v = v_1 + v_2 + v_3$

i-v Eqs.: $i = C_1 \dfrac{dv_1}{dt} \quad i = C_2 \dfrac{dv_2}{dt} \quad i = C_3 \dfrac{dv_3}{dt}$

Differentiating the KVL equation in time and substituting from capacitor i-v characteristics equations, we get:

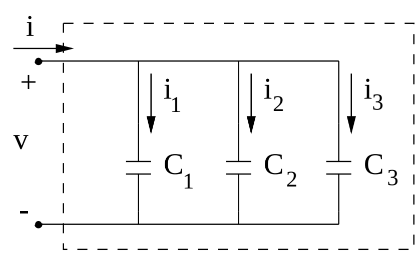
$$\frac{dv}{dt} = \frac{dv_1}{dt} + \frac{dv_2}{dt} + \frac{dv_3}{dt} = \frac{i}{C_1} + \frac{i}{C_3} + \frac{i}{C_3}$$

$$i = \left(\frac{1}{C_1} + \frac{1}{C_3} + \frac{1}{C_3}\right)^{-1} \frac{dv}{dt}$$

which is the i-v characteristics of the subcircuit. This is similar to the i-v characteristics of a single capacitor, $i = C_{eq} dv/dt$. Thus, **a set of capacitors in series reduce to one capacitor with a value of**

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + ...$$

Capacitors in Parallel



KCL: $i = i_1 + i_2 + i_3$

KVL: capacitors in parallel all have voltage v

i-v Eqs.: $i_1 = C_1 \dfrac{dv}{dt} \quad i_2 = C_2 \dfrac{dv}{dt} \quad i_3 = C_3 \dfrac{dv}{dt}$

Substituting from capacitor i-v characteristics equations in KCL, we get:

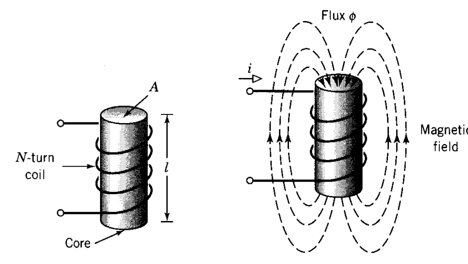
$$i = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + C_3 \frac{dv}{dt} = (C_1 + C_2 + C_3) \frac{dv}{dt}$$

This is similar to the i-v characteristics of a single capacitor, $i = C_{eq} dv/dt$. Thus, **a set of capacitors in parallel reduce to one capacitor with a value of**

$$C_{eq} = C_1 + C_2 + C_3 + ...$$

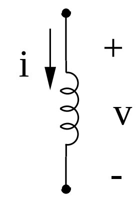
Inductor

A typical inductor is made of a a wire wrapped around a core of magnetic material. As current flows through the wire, a magnetic field is produced. As such, an inductor can store energy in the form of magnetic field.



(a) Essential parts of an inductor *(b)* Flux and magnetic field

The *i*-*v* characteristics equation for an <u>ideal</u> inductor is given by Faraday's Law (wire in an ideal inductor has no resistance):



$$v = L\frac{di}{dt} \qquad \text{Inductance } L\text{, Unit: Henry (H)}$$

using passive sign convention.

Energy stored in an inductor, W, can be found from the definition of electric power:

$$\frac{dW}{dt} = P = vi$$

$$W(t) - W(t_0) = \int_{t_0}^{t} P dt' = \int_{t_0}^{t} vidt' = \int_{t_0}^{t} Li\frac{di}{dt'}dt' = \frac{1}{2}L[i^2(t) - i^2(t_0)]$$

Since at $t_0 \to -\infty$, the current through and energy stored in the inductor are zero, we get:

$$W(t) = \frac{1}{2}Li^2(t)$$

Note: Unless the current waveform in the inductor is specified, we need an initial condition (current at some time t_0) in order to calculate current, voltage, and power of an inductor. (The condition of $i(t_0 \to -\infty) = 0$ is not usually useful.)

For inductors, the *i*-*v* characteristics equation $v = Ldi/dt$ implies that if there were a discontinuity in the current waveform, an infinite amount of voltage should appear across the inductor. Therefore, **the current waveform in an inductor should be continuous.** This means that if a switch is thrown at time t_0, we should have $i(t_0^-) = i(t_0^+)$. Note the inductor voltage waveform can have a discontinuity.

Inductors in series & parallel

Following similar procedure as that used for capacitors in series and in parallel, one finds that a set of inductors in series or in parallels reduce to one inductor with the value:

Inductors in Series: $L_{eq} = L_1 + L_2 + L_3 + ...$

Inductors in Parallel: $\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + ...$

Observation on circuits containing capacitors and inductors

1. Circuits containing one or more capacitors and/or inductors are called dynamic circuits. Voltage and currents in dynamic circuits are, in general, functions of time, *i.e.,* their value change as time progresses.

2. All analysis method (KVL, KCL, circuit reduction, node-voltage and mesh current methods) can be applied to dynamic circuits with no modification.

3. Application of the analysis methods to a dynamic circuit results in a set of linear differential equations (as opposed to a set of algebraic equations for resistive circuits). Solution of these equations requires initial conditions–one for each capacitor or inductor.

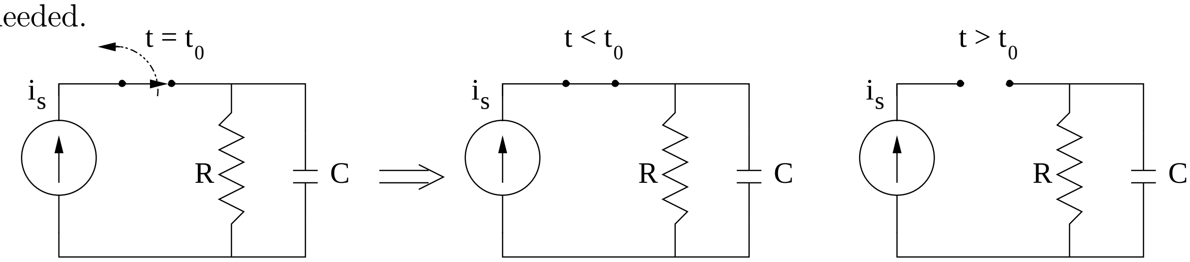
4. Two-terminal subcircuits containing capacitors and/or inductors cannot be reduced to Thevenin or Norton forms.

Switched Circuits

Time dependent circuits either have a time-dependent source and/or have a source which is "switched" on or off at some time. Most circuits have both, like a circuit attached to the wall outlet with a switch. The voltage source in the circuit is a time-dependent voltage (sinusoidal voltage with a frequency of 60 Hz). When the switch is off, the voltages and currents in the is zero. After the switch is thrown, the voltages and currents rise from zero and after some time start to follow the source voltage waveform in a steady manner and forget that they started from zero values. The evolution of circuit voltages and current in time, thus, can be divided into two distinct time scales. 1) A transient response starting with the throw of the switch and 2) A steady response to the time dependent sources (some circuit theory books call this "AC steady state"). We will see shortly that the transient response of the circuit is completely set by the circuit elements themselves and not by the "source." This transient response also "dies" away after some prescribed time.

Because of these two distinct responses of a time-dependent circuit, transient and steady responses can be studied separately. We will first examine the transient response of circuits after a switch is thrown. We use DC sources for this case as it will make the mathematics much simpler. We will later examine the "steady" response of circuits to time-dependent sources.

An example of a switched circuit is shown below. It is usually assumed that the circuit has been in the first state ($t < t_0$) for a long time and the response of the circuit for $t > t_0$ is needed.



The solution to the circuit for $t > t_0$ leads to a set of linear differential equations. These equations require a set of initial conditions (one for each capacitor and inductor). These initial conditions are found by solving the circuit for $t < t_0$ and using the "no-jump" conditions for voltages across capacitors and currents in inductors at the switching time. So, the procedure for solving switched circuits with DC sources is:

1. Solve DC steady-state case for $t < t_0$. Determine $v_c(t_0^-)$ and $i_L(t_0^-)$.
2. Use no-jump conditions to find the initial condition for the second circuit: $v_c(t_0^+) = v_c(t_0^-)$ and $i_L(t_0^+) = i_L(t_0^-)$.
3. Solve the time dependent circuit for $t > t_0$.

DC Steady-State Analysis

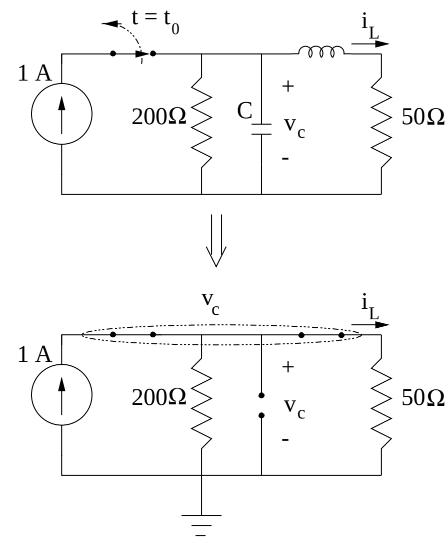
In a dynamic circuit with DC sources, voltages and currents reach constant values after "long time." (We will find how long is a "long time" shortly). This state of the circuit is called "DC Steady State." By definition, all time derivatives vanish at DC steady state condition:

For a capacitor, $i = C\dfrac{dv}{dt} = 0$ (for any v). Thus, **a capacitor acts as an open circuit in DC steady State.**

For an inductor, $v = L\dfrac{di}{dt} = 0$ (for any i). Thus, **an inductor acts as a short circuit in DC steady State.**

Therefore, to analyze a circuit in DC steady state condition, we replace capacitors with open circuit and inductor with short circuit and proceed to solve the resulting "resistive" circuit.

Example: This circuit has been in DC steady state and the switch is opened at time $t = t_0$. Find the initial conditions for the dynamic circuit for $t > t_0$.



First, we mark v_C and i_L on the original circuit (those are our initial conditions). We draw the circuit for $t_0 < t$ (closed switch) and replace the capacitor with an open circuit and the inductor with a short circuit (mark them clearly) and put v_C and i_L on the circuit. We then proceed to solve the "resistive" DC steady state circuit.

The resulting circuit is current divider and i_L and v_C can be found readily from current-divider formulas. Alternatively, using node-voltage method, we have:

$$\frac{v_C - 0}{50} + \frac{v_C - 0}{200} - 1 = 0$$

$$4v_C + v_C - 200 = 0 \qquad \rightarrow \qquad v_C = 40 \text{ V}$$

$$i_L = \frac{v_C - 0}{50} = 0.8 \text{ A}$$

Note that since the circuit is in DC steady state, values of v_C and i_L are constant in time. Using the no-jump conditions for v_C and i_L, we can find the initial conditions for the time-dependent circuit:

$$v_C(t = t_0^+) = 40 \text{ V} \qquad \rightarrow \qquad i_L(t = t_0^+) = 0.8 \text{ A}$$