

## Circuit Variables

Scientific investigation of static electricity was done in late 1700's and Coulomb is credited with most of the discoveries. He found that electric charges have two attributes: amount and polarity. There are two type of charges – opposite charges attract and similar polarity ones repel each other. Charge polarity is indicated by positive and negative signs because positive and negative charges cancel each other when brought together. As a results, the electric charge can be described by an algebraic number,  $q$ , with units of Coulomb (C).

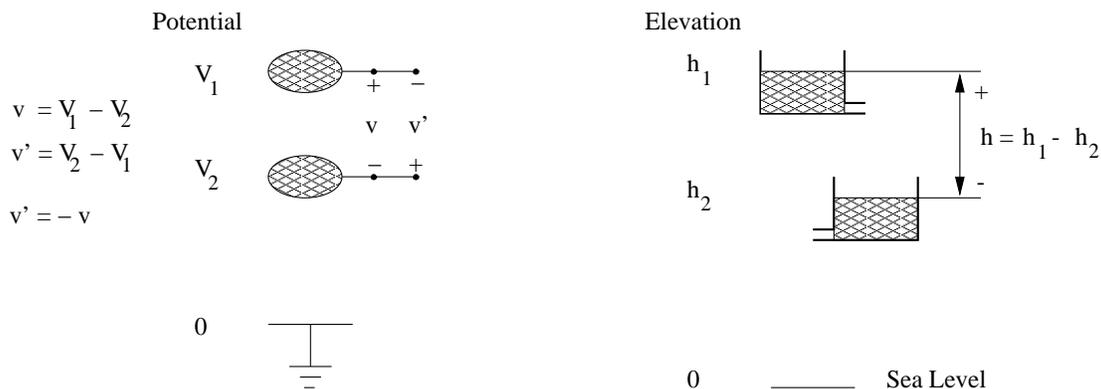
Because opposite charges attract each other, energy is expended to separate them from each other. This energy is stored in the electric field between the two “reservoir” of separated charges and is recovered when the charges are allowed to come together. The stored energy per unit charge is called the “voltage” or potential difference between the two reservoir of charges:

$$v = \frac{dW}{dq} \quad \text{Unit: volt (V = J/C)}$$

Note that we need two reservoir of charges. So voltage is between two points. We also use + and – signs to indicate the direction for measuring  $v$ . From definition of voltage above,  $w$  is energy needed to move a positive charge from – reservoir to + reservoir.

One can define a reference point for measuring voltages (typically shown as ground). The voltage between any point and this reference point is call the potential of that point. It is always assumed that + is at the point and the – sign is at the reference point. Therefore, there is no need to indicate + and – signs for potential. Voltage between two points is the difference between the potential of the two points (see figure).

Voltage between two charge reservoirs is analogous to height difference between two fluid reservoir and the same way, the potential of each point is analogous to its elevation compared to some reference (*e.g.*, sea level).



If we connect the charge reservoirs, electric charges flow from one to other. The rate of the charge flow through a specific area is called the electric current:

$$i = \frac{dq}{dt} \quad \text{Unit: Ampere (A = C/s)}$$

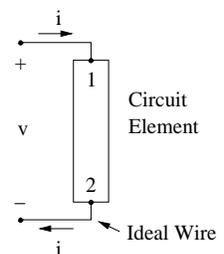
with the current flowing in the direction of the charge flow (it means that a positive current is associated with the flow of positive charge).

In principle, electric charges generate an electric field and motion of the charged particles (current) generates a magnetic field. This electromagnetic field interacts with all charges and affect them. The behavior of such a system is described by Maxwell's equation. Solution of Maxwell's equations, however, is difficult and not needed expect for some cases (propagation of electromagnetic wave and light, antennas, etc.)

In most relevant engineering cases the problem can be greatly simplified by noting that electric charges preferentially flow through a conductor (or a semiconductor) as opposed to vacuum, air, or any insulator. In this case, the system can be described as a circuit containing "circuit elements" and "connecting ideal wires." Circuit theory is the scientific discipline that describes behavior of circuits be and is built upon the following assumptions:

1. All of the electromagnetic phenomena occurs inside each circuit element. They communicate with the outside world only through the voltage across and current that particular element.
2. Circuit elements are connected to each other with ideal wires that do not impede flow of charge. They can be stretched (making them longer or shorter for example) without any effect on the circuit.
3. Net amount of charge cannot be accumulated in any circuit element or any location in the circuit. If a net charge of  $q$  enters a circuit element, the same amount of charge should leave the element. This means: (a) a circuit element should have at least two terminals, (b) Because current travels through the system at a good fraction of speed of light, we can safely assume that the total current entering a circuit element is exactly equal to the current leaving that element at any instant time.

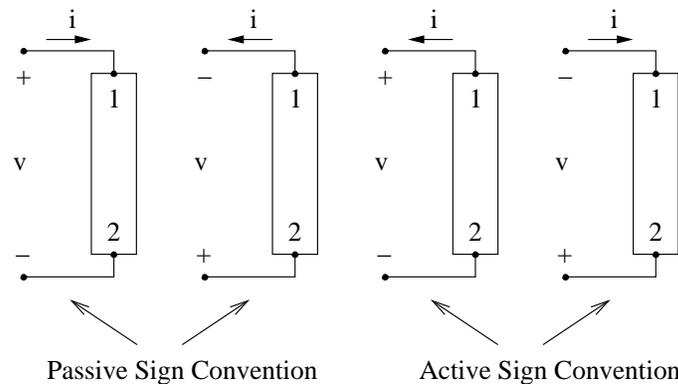
A note about current flow in a conductor. Consider a series of beads on a wire. If another bead is added to the left end of wire, all of the bead move one step to the right and one bead will fall off the right end. Similarly, when an electron enters one end of a wire, electrons in the system move slightly and another electron will leave the other end of the wire. As such, while electrons do not move rapidly in a conductor, the current in the wire propagates at a good fraction of speed light.



In the context of circuit definition above, the important physical quantities are current and voltages (and electric power). These are the **circuit variables** that form the basis for communication between circuit elements.

The value of voltage between two points,  $v$ , is incomplete unless the  $+$  and  $-$  signs are assigned to the two points. Similarly, the value of current is incomplete unless a reference current direction is assigned. However, as  $v$  and  $i$  are algebraic numbers, the assigned position of  $+$  and  $-$  for voltage and direction for the current is arbitrary – the sign of  $v$  and  $i$  flips according to this choice.

Internal of each circuit element impose a relationship between the current flowing in the element and the voltage across that element. This relationship is called the element law or  $i$ - $v$  characteristics. While the choice of reference directions for current through and voltage across an element is arbitrary, the element law or  $i$ - $v$  characteristics of an element depends on the reference directions. To see this point, note that there exists four possible choices for voltage and current directions in a two-terminal circuit element:



In two left cases, the current direction is marked such that it flows from  $+$  to  $-$  signs. This case is called the passive sign convention. In the two right cases, current flows from  $-$  to  $+$  signs. This case is referred to as active sign convention. Obviously, the  $i$ - $v$  characteristic will be different in each case as the sign of current or voltage is switched. To resolve this confusion, we follow this convention:

1.  $i$ - $v$  characteristics of two-terminal element are written assuming passive sign convention.
2. Use passive sign convention when marking current and voltages in a circuit as much as possible.

So, in marking references for voltages and currents in the circuit, it is best to arbitrarily choose either the voltage or current reference direction and then use passive sign convention

to mark the other variable. Note that it may not be possible or practical to mark every element using passive sign convention. In this case, *A good rule is to mark every element except voltage and current sources using passive sign convention.*

**Electric power produced or absorbed in an element:**

Power:  $P = \frac{dW}{dt}$                       Unit: Watt (W=J/s).

Since  $v = dW/dq$  and  $i = dq/dt$  (both total derivatives), then:

$$P = \frac{dW}{dt} = \frac{dW}{dq} \times \frac{dq}{dt}$$

$$P = i \times v$$

Using the definition of the voltage, one finds that if we use **passive sign convention**:

$P < 0$                       Element is producing power

$P > 0$                       Element is absorbing power

Obviously, if we use **active sign convention**:

$P < 0$                       Element is absorbing power

$P > 0$                       Element is producing power

**Example:** Find the power absorbed or supplied by elements A and B if  $i = 25$  A and  $v = 120$  V.

For element A,

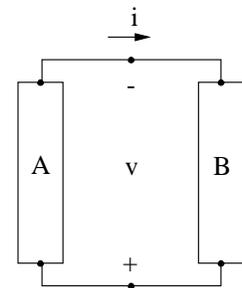
$$P = vi = 25 \times 120 = 3,000 \text{ W} = 3 \text{ kW}$$

Since element A reference directions follow passive sign convention and  $P >$ , element A is absorbing 3 kW of power.

For element B,

$$P = vi = 25 \times 120 = 3,000 \text{ W} = 3 \text{ kW}$$

Since element B reference directions follow active sign convention and  $P >$ , element B is supplying 3 kW of power.



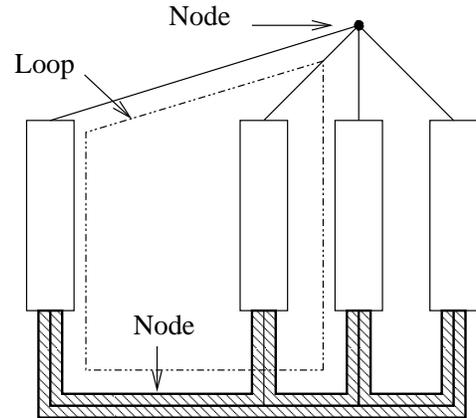
# KIRCHHOFF LAWS

## Definitions

A **circuit** consists of circuit elements attached to each other with ideal wires or connectors.

**Node:** A node is a point in the circuit that is connected to two or more circuit elements with ideal wires.

**Loop:** A loop is a closed path in a circuit through at least two circuit elements which return to starting node without passing through any node twice.



Note that a node should not be taken literally as a geometric point on a specific circuit diagram as the ideal wires connecting the elements can be moved and stretched. For example, any point on the ideal wires connected to the four elements in the bottom of the figure is a node (as highlighted in the figure). All these points represent the same node.

**Kirchhoff Current Law (KCL):** KCL follows from our assumption that no net charge can be accumulated at any point in the circuit (or in a node):

*Sum of currents entering a node is equal to sum of currents leaving a node.*

Alternatively, as a current leaving a node can be replaced with a current entering a node with opposite sign, KCL can be stated as:

**Kirchhoff Current Law (KCL):** *Algebraic sum of currents entering a node is zero.*  
or *Algebraic sum of currents leaving a node is zero.*

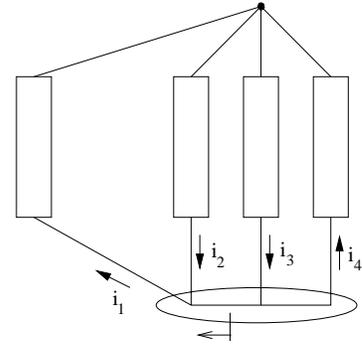
I use the last version of KCL in these notes as this version is also used in the node-voltage method.

## How to apply KCL

1. You should have identified the nodes and mark currents and their reference directions on it.
2. Draw a closed path around the node and mark a starting a point.
3. Go around the path in the clockwise direction. Whenever you pass a wire, write down the current with a + sign if exiting and a - sign if entering.
4. Stop when you are back to the starting point (and write = 0).

**Example:** Write KCL for the marked node.

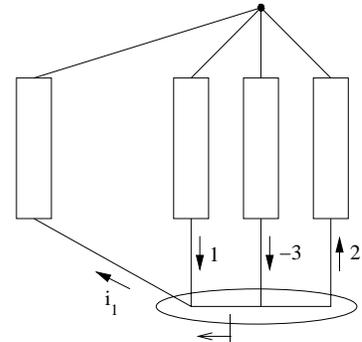
$$+i_1 - i_2 - i_3 + i_4 = 0$$



**Example:** Find  $i_1$

$$+i_1 - (1) - (-3) + (2) = 0$$

$$i_1 = -4 \text{ A}$$



**Kirchhoff Voltage Law (KVL):** KVL follows from the definition of the voltage between points. Voltage is defined as the amount energy to move unit charge from one point to another. Zero net energy should be expended if a charge  $q$  is moved around a closed loop and is returned back to its original position, *i.e.*,

$$\Sigma \Delta W = q \Sigma \Delta v = 0 \quad \rightarrow \quad \Sigma \Delta v = 0$$

**Kirchhoff Voltage Law (KVL):** Algebraic sum of all voltages around any closed path in a circuit is zero.

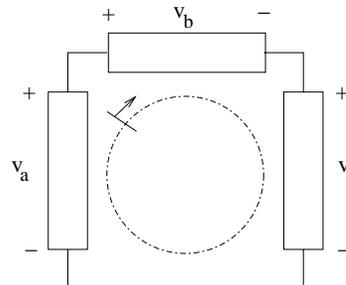
or *Sum of voltage drops around a loop - Sum of voltage rises around a loop = 0.*

### How to apply KVL

1. You should have marked voltages and their reference directions on it.
2. Draw the loop and mark a starting a point.
3. Go around the path in the clockwise direction. Whenever you go over a circuit element, write down the voltage across element with a + sign if entering the + terminal of an element and a - sign if entering the - terminal of an element.
4. Stop when you are back to the starting point (and write = 0).

**Example:** Write KVL for the marked loop.

$$+v_b + v_c - v_a = 0$$



KVLs and KCLs are constraints on circuit variables which arise because of the circuit arrangement (attachment of connection wires). In addition, internal of each circuit element impose a relationship between the current flowing in the element and the voltage across that element (element Laws or  $i$ - $v$  characteristics). Combination of these two set of constraints results in a unique set of values for the circuit variables (currents and voltages).

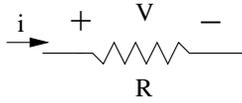
In a circuit with  $E$  elements, there are  $2E$  circuit variables ( $i$  and  $v$  for each element). We need  $2E$  equations to find these circuit variables. If the circuit has  $N$  nodes, we can write  $N - 1$  KCL equations (KCL on the  $N^{th}$  node is exactly the sum of KCL on the other  $N - 1$  nodes). We can also write  $E - N + 1$  independent KVLs and  $E$   $i$ - $v$  characteristic equations:

No. of KCL equations:	$N - 1$	
No. of KCV equations:	$E - N + 1$	
No. of $i$ - $v$ characteristics:	$E$	
	$2E$	Total

If the  $i$ - $v$  characteristic is a linear relationship between  $i$  and  $v$ , the element is a linear element. If a circuit is made of linear element, the resulting set of  $2E$  equations in  $2E$  variables for a linear algebraic set of equations. In this course, we only use linear elements, thus, the term “linear circuit theory.”

## Linear Circuit Elements

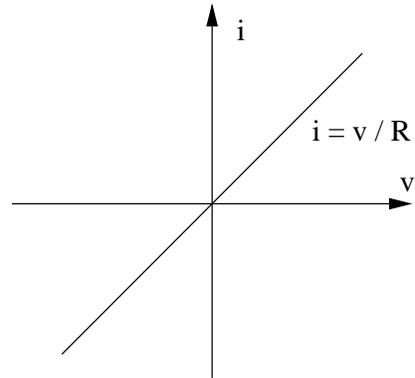
### Resistor



$i$ - $v$  characteristic:  $v = Ri$

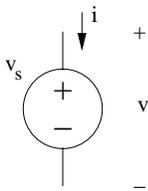
Resistance  $R$ , Unit Ohm ( $\Omega$ )

Conductance  $G = 1/R$ , Unit Siemens (S)

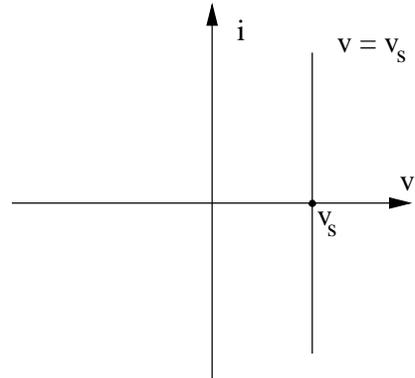


$$P = vi = Ri^2 \quad \rightarrow \quad P > 0 : \text{Resistor always absorbs power}$$

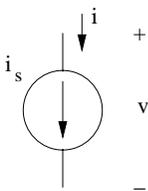
### Independent Voltage Source (IVS)



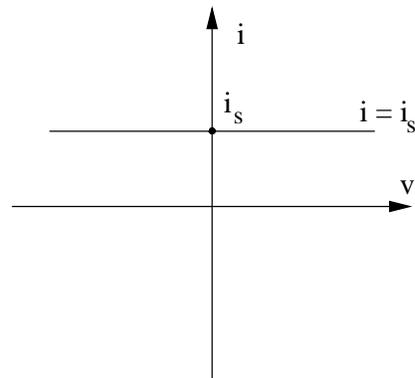
$i$ - $v$  characteristic:  $v = v_s$  for any  $i$



### Independent Current Source (ICS)



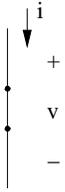
$i$ - $v$  characteristic:  $i = i_s$  for any  $v$



### Some Observations:

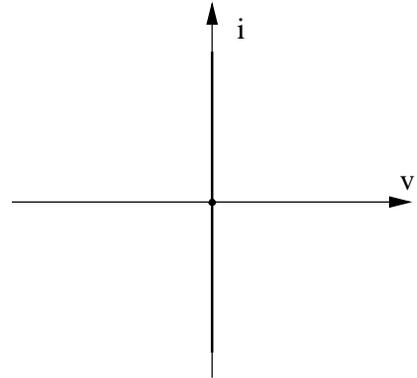
1.  $i$ - $v$  characteristics (element laws) of linear circuit element are lines in  $i$ - $v$  plane.
2.  $i$ - $v$  characteristics of IVS and ICS are independent of sign convention (*e.g.*, equally correct for both passive and active sign conventions).
3. Resistors always absorb power. IVS and ICS can either absorb or supply power.

## Short Circuit

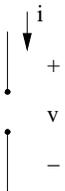


$i$ - $v$  characteristic:  $v = 0$  for any  $i$

Note that a “short circuit” element is a special case of a resistor (with  $R = 0$ ) or an ideal voltage source (with  $v_s = 0$ ).

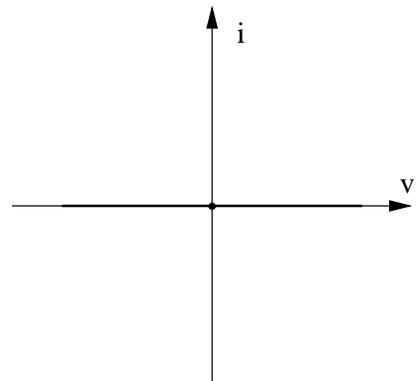


## Open Circuit



$i$ - $v$  characteristic:  $i = 0$  for any  $v$

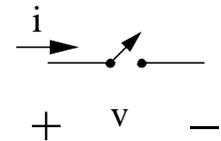
Note that an “open circuit” element is a special case of a resistor (with  $R \rightarrow \infty$ ) or an ideal current source (with  $i_s = 0$ ).



## Switch

$i$ - $v$  characteristic:

- Switch open:  $\rightarrow$  Open circuit  $i = 0$  for any  $v$
- Switch closed:  $\rightarrow$  Short circuit  $v = 0$  for any  $i$



## Procedure Circuit Analysis using KVL and KCL

1. Note how you can calculate problem unknown (*e.g.*, power dissipation in an element) from the circuit variables.
2. Go through the circuit in an orderly fashion (*e.g.*, from left to right, top to bottom). Take each element and
  - 2a) Identify nodes (terminals of each element should be connected to a node.)
  - 2b) Assign voltages and currents and their reference directions to each element. Use passive sign convention.
  - 2c) Identify circuit variables. This will tell you how many equations you have to write.
  - 2d) Write down  $i$ - $v$  characteristics equation.
3. If you have  $N$  nodes, write  $N - 1$  KCLs.
4. Calculate no. of KVLs you need:  $N_{KVL} = 2E - (E) - (N - 1) = E - N + 1$  (where  $E$  is no. of elements). Note that as some element laws (*e.g.*, IVS, ICS) result in trivial equations for some circuit variables, it is better to count the circuit unknowns from step 2 above and use  $N_{KVL} = N_{unknowns} - N_{element\ laws} - N_{Nodes} + 1$ .
5. Choose  $N_{KVL}$  loops and write KVLs. Choose loops that go through the smallest number of elements.
6. Solve the system of equations and find circuit variables.
7. Check your solution by applying your solution to some KVL and KCL (specially KVL on loops you have not considered).
8. Find problem unknown, if any, from the circuit variables.

**Note:** It is not always possible or practical to assign passive sign convention to all elements. If so, note that  $i$ - $v$  characteristics of IVS and ICS are independent of sign convention. You do NOT need to use passive sign convention for these elements. But be careful if you need to find power supplied or absorbed by IVS or ICS.

**Note:** You can readily reduce the number of equations to solve by half if you substitute  $i$ - $v$  characteristics equations in KVLs and KCLs to get a set of  $E$  equations in either voltages or currents (or a combination of both). In fact, after you are fluent in using the above procedure, you only need to mark either the current or voltage for an element and use the element law to directly write the other circuit variable on the circuit diagram. Then, write only KCLs and KVLs and solve.

**Example:** Find  $i_1$ .

Following the procedure, we first mark the circuit variables (11 unknowns:  $v_0, v_1, v_2, v_3, v_4, i_0, i_1, i_2, i_3, i_4, i_5$ ) and identify the nodes, and write the  $i$ - $v$  characteristics equations:

$$\begin{aligned} v_0 &= 25i_0 & v_1 &= 70i_1 \\ v_2 &= 5i_2 & v_3 &= 10i_3 \\ v_4 &= 8i_4 \end{aligned}$$

Note that we have 6 elements so we have 12 circuit variables. However, the IVS element law specifies the voltage across its terminal, so we have 11 circuit variables which have to be found.

The circuit has four nodes. Since we need to write KCL only in  $N - 1$  nodes, we choose not write KCL in the bottom node. the KCLs are:

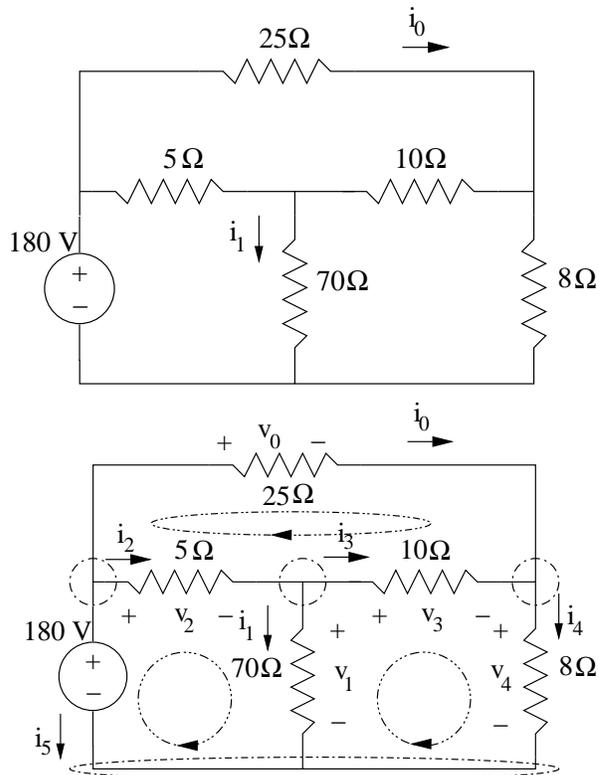
$$\begin{aligned} i_0 + i_2 + i_5 &= 0 \\ i_3 + i_1 - i_2 &= 0 \\ i_4 - i_3 - i_0 &= 0 \end{aligned}$$

We need to write  $N_{KVL} = 11 - 5 - 3 = 3$  KVLs. Choosing 3 loops with smallest number elements, we get:

$$\begin{aligned} +v_0 - v_3 - v_2 &= 0 \\ +v_2 + v_1 - (180) &= 0 \\ +v_3 + v_4 - v_1 &= 0 \end{aligned}$$

Above are eleven equations in eleven unknowns that can be solved. The numbers of equations to be solved can be halved by using  $i$ - $v$  characteristics equations to substitute in KVLs and KCLs. For example, if we substitute for voltages from  $i$ - $v$  characteristics equations in KVLs, we get 6 equations in 6 unknown currents:

$$\begin{aligned} i_0 + i_2 + i_5 &= 0 & 25i_0 - 10i_3 - 5i_2 &= 0 \\ i_3 + i_1 - i_2 &= 0 & 5i_2 + 70i_1 &= 180 \\ i_4 - i_3 + i_0 &= 0 & 10i_3 + 8i_4 - 70i_1 &= 0 \end{aligned}$$



**Note:** As can be seen, even relatively small circuits results in a large number equations to be solved. Several techniques for reducing the number of equation to be solved are introduced next.