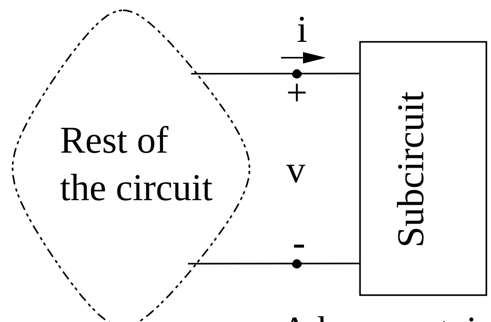
Circuit Reduction Techniques

Combination of KVLs, KCLs, and i-v characteristics equations result in a set of linear equations for the circuit variables. While the above set of equation is complete and contains all necessary information, even small circuits require a large number of simultaneous equations to be solved as seen previously. We learned that we can use i-v characteristics equations when we are marking the circuit variables and reduce the number of equations to be solved to only number of KCLs and KVLs. We will learn later two methods, node-voltage and mesh-current, which reduce the number of equations to be solved further to either number of KCLs or number of KVLs. This is the best we can do in this direction.

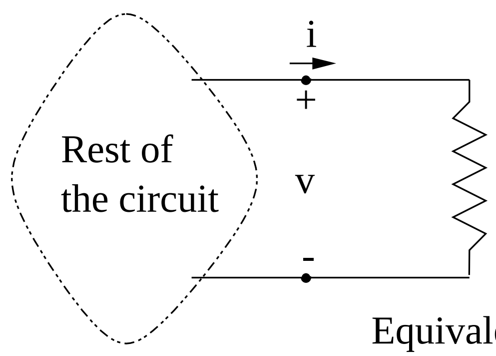
As it is easier to solve smaller sets of equations (*e.g.,* it is easier to solve two sets of two equations in two unknown as compared to a set of 4 equations in 4 unknowns), one can break up the circuit into smaller pieces and solve each individually and assemble back the whole circuit. We can use this principle to combine circuit elements and make a much smaller circuit. These techniques are described below.

Recall that we are using lumped circuit elements, *i.e.,* circuit elements communicate to outside world and other circuit elements only through i and v. Conversely, the outside world (the rest of the circuit) communicate with the circuit element through i and v. This means, for example, that a resistor in a circuit is viewed by the rest of the circuit as a "black box" with an i-v characteristics of $v = Ri$. The rest of the circuit does not know what is inside the box. In fact, we can replace the resistor with any black box (containing whatever) with the same i-v characteristics of $v = Ri$ and the rest of the circuit behaves exactly the same.

Alternatively, if a black box containing many circuit elements is attached to a circuit and has an i-v characteristics of $v = 5i$, we can replace this black box with a 5 Ω resistor with no change in the circuit behavior.



A box containin some elements



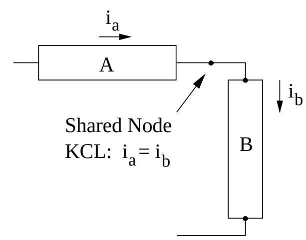
Equivalent Subcircuit

This observation allows the circuit to be divided into two or many parts and each solved independently. We define a box containing several element a **subcircuit** or a **device**. The above figure shows a two-terminal device or subcircuit. Note that in many circuit theory text book (including our textbook) circuit and subcircuit are used interchangeably.

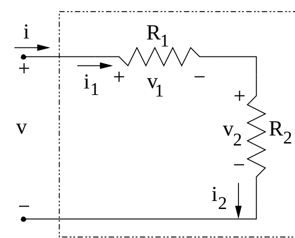
Subcircuits play an important role in linear circuit theory. Thevenin theorem states that any subcircuit containing linear circuit element has an i-v characteristics of $Av + Bi = C$ (where A, B, and C are constants) and it can be reduced to a subcircuit containing at most two linear circuit elements (Thevenin and Norton Forms). We will discuss Thevenin Theorem later. Below, we explore subcircuits in the context of elements that are in series or in parallel. In each case, we find the i-v characteristics of the subcircuit and use that to find the equivalent element.

Elements in Series

Two elements are called in series if they and only they share a common node. Alternatively, series-connected elements carry the same current.



Two Resistors in Series



$$
\begin{aligned}
\text{KCL:}\quad & i = i_1 = i_2 \\
\text{KVL:}\quad & -v + v_1 + v_2 = 0 \\
i\text{-}v\text{:}\quad & v_1 = R_1 i_1 = R_1 i \\
& v_2 = R_2 i_2 = R_2 i
\end{aligned}
$$

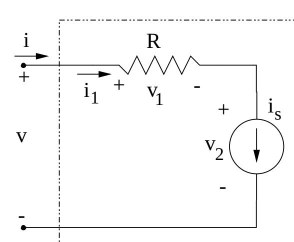
Substituting from i-v characteristics equations in KVL, we get

$$
\begin{aligned}
v &= R_1 i + R_2 i = (R_1 + R_2) i \\
v &= R_{eq} i \qquad R_{eq} = R_1 + R_2
\end{aligned}
$$

So, a subcircuit containing two resistors in series has an i-v characteristics of the form $v = R_{eq} i$ and is equivalent to a resistor, $R_{eq} = R_1 + R_2$.

The above can be easily extended: k resistors in series are equivalent to one resistor with $R_{eq} = \Sigma_{j=1}^{k} R_j$.

A Resistor in Series with a Current Source

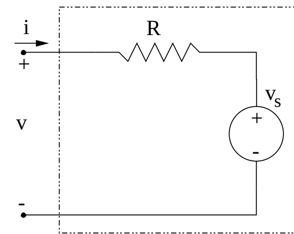


$$
\begin{aligned}
\text{KCL:}\quad & i = i_1 = i_s \\
\text{KVL:}\quad & -v + v_1 + v_2 = 0 \qquad \rightarrow \qquad v = R_1 i_s + v_2
\end{aligned}
$$

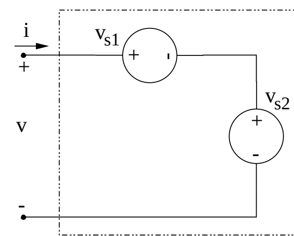
The i-v characteristics of ICS states that its current is i_s, independent of its voltage (v_2). Above equations show that the i-v characteristics of subcircuit is $i = i_s$ and is independent of voltage $v = v_1 + v_2$ (as value of v_1 can be anything). The equivalent subcircuit is an independent current source with strength i_s.

A Resistor in Series with a Voltage Source

This is the Thevenin form and cannot be reduced further.



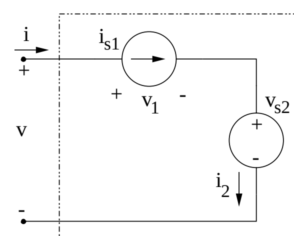
A Voltage Source in Series with a Voltage Source



KVL: $\quad -v + v_{s1} + v_{s2} = 0 \quad \rightarrow \quad v = v_{s1} + v_{s2}$

The *i-v* characteristics of IVS states that their voltages are v_{s1} and v_{s2}, respectively, independent of their currents. Above equation shows that the *i-v* characteristics of subcircuit is $v = v_{s1} + v_{s2}$ and is independent of current i. The equivalent subcircuit is an independent voltage source with strength $v_{s1} + v_{s2}$ (algebraic sum of two). Note that it is prudent to use KVL to find the strngeth of the equivalent source.

A Voltage Source in Series with a Current Source



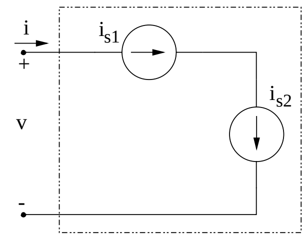
KCL: $\quad i = i_{s1} = i_2$

KVL: $\quad -v + v_1 + v_{s2} = 0 \quad \rightarrow \quad v = v_1 + v_{s2}$

The *i-v* characteristics of ICS states that its current is i_{s1}, independent of its (v_1). Above equations show that the *i-v* characteristics of subcircuit is $i = i_{s1}$ and is independent of voltage $v = v_1 + v_{s2}$ (as value of v_1 can be anything). The equivalent subcircuit is an independent current source with strength i_{s1}.

Note: A current source in series with any element reduced to a current source. Series element all have the same current and the current source requires the current through to be equal to its strength.

A Current Source in Series with a Current Source



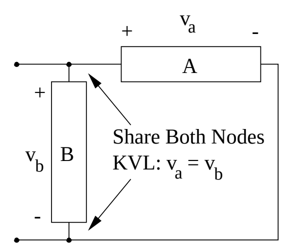
KCL: $i = i_{s1} = i_{s2}$

A current source in series with any element reduces to a current source. However, in the case of two current sources in series, KCL requires $i_{s1} = i_{s2}$. Thus, two current sources can be attached in series only if $i_{s1} = i_{s2}$. If so, the equivalent subcircuit is an independent current source with strength $i_s = i_{s1} = i_{s2}$.

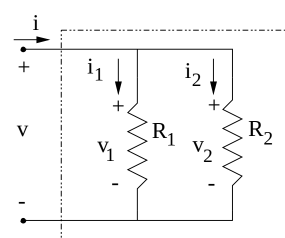
This constraint in allowable circuit configuration arises because we are dealing with idealized circuit elements. We will discuss real sources later and will see that two real or practical current sources can be attached in series even if they have different strength (although the result may be a lot of sparks and two burnt out current sources!)

Elements in Parallel

Two element are called in parallel if they share both nodes. Alternatively, parallel-connected elements have the same voltage.



Two Resistors in Parallel



KCL: $-i + i_1 + i_2 = 0$

KVL: $v = v_1 = v_2$

i-*v*: $v_1 = R_1 i_1 \qquad v_2 = R_2 i_2$

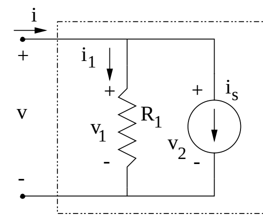
Substituting from *i*-*v* characteristics equations in KCL, and using $v = v_1 = v_2$, we get

$$
\begin{aligned}
i &= \frac{v}{R_1} + \frac{v}{R_2} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right) \times v \\
i &= \frac{v}{R_{eq}} \qquad \rightarrow \frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}
\end{aligned}
$$

So, a subcircuit containing two resistors in parallel has an *i*-*v* characteristics of the form $v = R_{eq} i$ and is equivalent to a resistor, $1/R_{eq} = 1/R_1 + 1/R_2$.

The above can be easily extended: k resistors in series are equivalent to one resistor with $1/R_{eq} = \Sigma_{j=1}^{k} 1/R_j$.

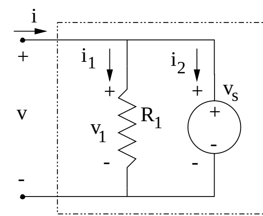
A Resistor in Parallel with a Current Source



This is the Norton form and cannot be reduced further.

We will show later that Norton and Thevenin forms are equivalent.

A Resistor in Parallel with a Voltage Source

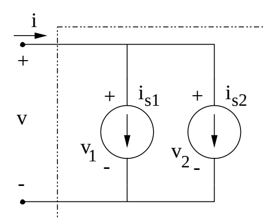


KCL: $-i + i_1 + i_2 = 0$

KVL: $v = v_1 = v_s$

The i-v characteristics of IVS states that its voltage is v_s, independent of its current, i_2. Above equations show that the i-v characteristics of subcircuit is $v = v_s$ independent of current i. The equivalent subcircuit is an independent voltage source with strength v_s.

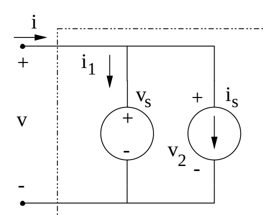
A Current Source in Parallel with a Current Source



KCL: $-i + i_{s1} + i_{s2} = 0 \qquad \rightarrow \qquad i = i_{s1} + i_{s2}$

The i-v characteristics of ICSs state that their currents are i_{s1} and i_{s2}, respectively, independent of their voltage, v. Above equations show that the i-v characteristics of subcircuit is $i = i_{s1} + i_{s2}$ independent of value of voltage v. The equivalent subcircuit is an independent voltage source with strength $i_s = i_{s1} + i_{s2}$.

A Current Source in Parallel with a Voltage Source



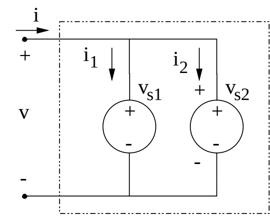
KCL: $-i + i_1 + i_2 = 0$

KVL: $v = v_s = v_2$

The i-v characteristics of IVS states that its voltage is v_s, independent of its current, i_1. Above equations show that the i-v characteristics of subcircuit is $v = v_s$ and is independent of current $i = i_1 + i_s$ (as value of i_1 can be anything). The equivalent subcircuit is an independent voltage source with strength v_s.

Note: A voltage source in parallel with any element reduces to a voltage source. Parallel elements all have the same voltage and the voltage source requires its voltage across to be equal to its strength.

A Voltage Source in Parallel with a Voltage Source



KVL: $v = v_{s1} = v_{s2}$

A voltage source in parallel with any element reduces to a voltage source. However, in the case of two voltage sources in parallel, KVL requires $v_{s1} = v_{s2}$. Thus, two voltage sources can be attached in parallel only if $v_{s1} = v_{s2}$. If so, the equivalent subcircuit is an independent voltage source with strength $v_s = v_{s1} = v_{s2}$. This constraint in allowable circuit configuration arises because we are dealing wit idealized circuit element. We will discuss real sources later.

Summary of Two-terminal Equivalent Subcircuits

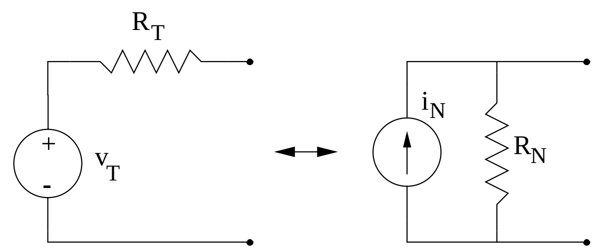
	Series	Parallel
R_1 and R_2	Resistor ($R_{eq} = R_1 + R_2$)	Resistor ($1/R_{eq} = 1/R_1 + 1/R_2$)
R and IVS (v_s)	Thevenin Form*	IVS (v_s)
R and ICS (i_s)	ICS (i_s)	Norton Form*
IVS (v_{s1}) and IVS (v_{s2})	IVS ($v_s = v_{s1} + v_{s2}$)	IVS ($v_s = v_{s1} = v_{s2}$)†
IVS (v_{s1}) and ICS (i_{s2})	ICS ($i_s 2$)	IVS (v_s)
ICS (i_{s1}) and ICS (i_{s2})	ICS ($i_s = i_{s1} = i_{s2}$)†	ICS ($i_s = i_{s1} + i_{s2}$)

* Thevenin and Norton forms are equivalent.

† Connection is allowed only if $v_{s1} = v_{s2}$ or $i_{s1} = i_{s2}$.

Thevenin and Norton Forms and Source Transformation

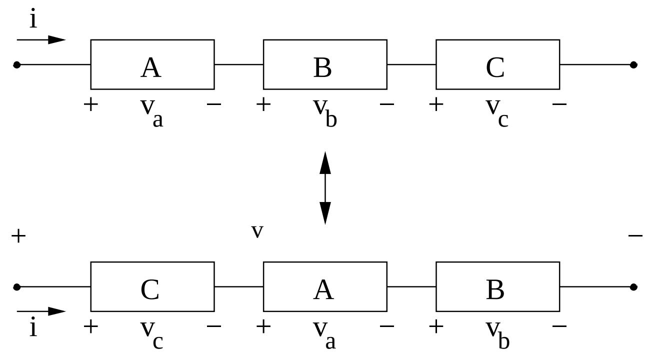
Thevenin and Norton forms are equivalent. One can replace one with the other. This is called "source" transformation and is helpful in rearranging other elements in the circuit and sometime arriving at more element being in series and parallel. Watch out for polarities of the IVS and the ICS and follow the diagrams on the left!



Equivalent if $R_T = R_N$ and $v_T = i_N R_T$

Three or More Element in Series

Position of elements in series can be interchanged without any effect on the circuit. The reason is that KCL ensures that all elements carry the same current i, the voltage across each element is uniquely set by its i-v characteristics and value of i, and the voltage over all of the elements, $v = v_a + v_b + v_c = v_c + v_a + v_b$ is the same if we interchange the position of the elements.



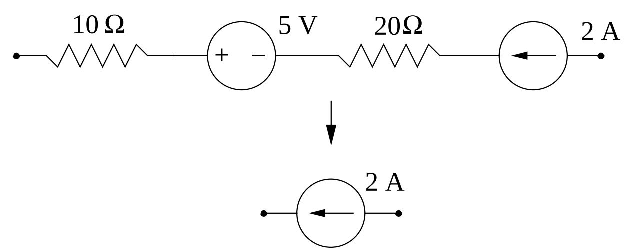
To simplify circuits with 3 or more elements in series:

1. Check if there is a current source. If so, all series elements can be replaced with a current source of the same strength. Note that if there are more than one current source, you should check for illegal connections of two current sources in series.

2. Rearrange elements and group resistors and voltage sources together. Replace resistors with a resistor, $R_{eq} = \Sigma R_j$ and voltage sources with a voltage source with strength $v_s = \Sigma v_{sj}$. It is prudent to use KVL to ensure that you get the correct algebraic sum of Σv_{sj}.

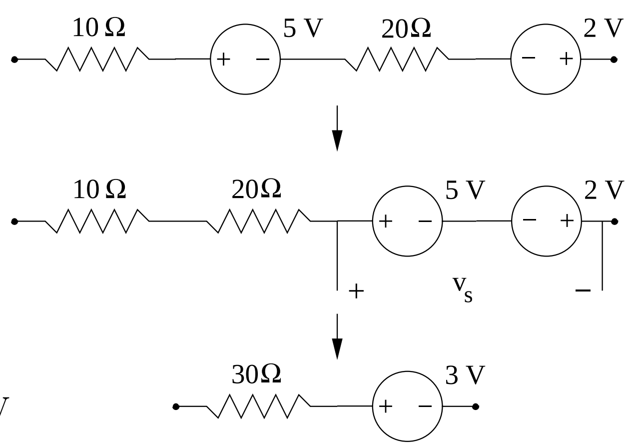
Example:

Four elements are in series.
One current source exists.
The equivalent element is a current source.



Example:

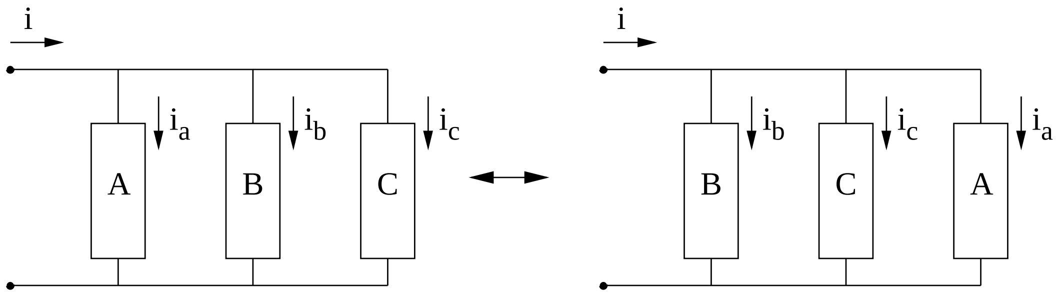
Four elements are in series.
No current source exists.
Group resistors and IVSs together.



$R_{eq} = 10 + 20 = 30\ \Omega$

KVL: $+5 - 2 - v_s = 0 \quad \rightarrow \quad v_s = 3$ V

Three or More Element in Parallel



Position of elements in parallel can be interchanged without any effect on the circuit because the ideal wires can be stretched without any impact on the circuit. (Imagine lifting element A out of the paper, moving it behind element C and putting it back on the paper.)

To simplify circuits with 3 or more elements in parallel:

1. Check if there is a voltage source. If so, all parallel elements can be replaced with a voltage source of the same strength. Note that if there are more than one voltage source, you should check for illegal connections of two voltages sources in parallel.

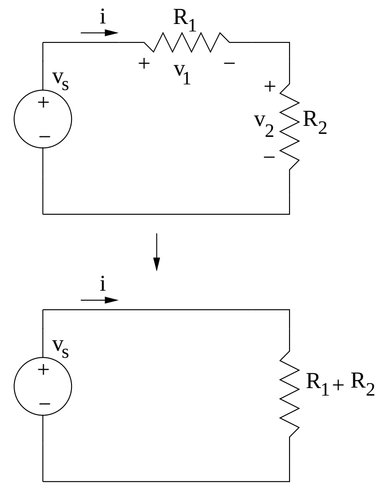
2. Rearrange elements and group resistors and current sources together. Replace resistors with a resistor, $1/R_{eq} = \Sigma\ 1/R_j$ and current sources with a current source with strength $i_s = \Sigma i_{sj}$. It is prudent to use KCL to ensure that you get the correct algebraic sum of Σi_{sj}.

Application of Circuit Reduction Techniques

Circuit reduction techniques are powerful methods to simplify the circuit as they reduce the number of elements (and therefore, the number of equations to be solved simultaneously). However, when several circuit element are combined, the circuit variables associated with those elements are lost in the process of transformation. In principle, one should solve the simplifies circuit and find the remaining circuit variables. Then, one should go back to the original circuit to find the "lost" circuit variables. Two examples below show this process. Because of this extra step, circuit reduction techniques do not always lead to simpler circuits and they should be used judiciously.

Example: Find v_2 if $R_1 = R_2 = 1$ kΩ and $v_s = 5$ V.

Resistors R_1 and R_2 are in series, so we can replace them with an equivalent resistor $R_{eq} = R_1 + R_2 = 2$ kΩ. The resultant circuit is shown. From the point of view of the rest of the circuit (*i.e.,* IVS) nothing has changed and so the current i remains the same. However, the circuit variables in the reduced part, v_1 and v_2 do not appear in the reduced circuit. To find v_2, we first solve the reduced circuit to find i:



$$\text{KVL:} \qquad -v_s + iR_{eq} = 0 \rightarrow \qquad i = \frac{v_s}{R_{eq}} = \frac{5}{2,000} = 2.5 \text{ mA}$$

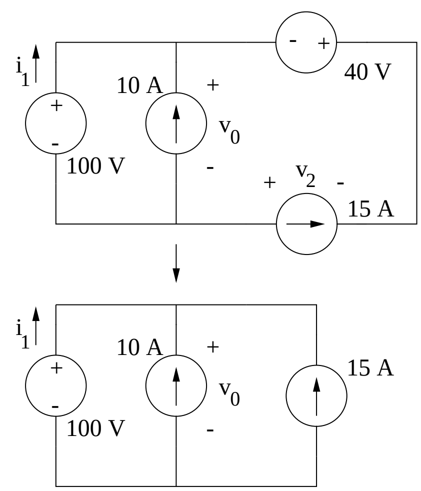
We then use the value of i in the original circuit to find v_2:

$$v_2 = R_2 i = 1,000 \times 2.5 \times 10^{-3} = 2.5 \text{ V}$$

Example: Find v_0 and i_1.

In this circuit, we have two sources in parallel and two sources in series. The problem unknowns are voltage and current of the sources that are in parallel. So, it is not prudent in the first step to combine them. Rather, we combine the two sources in series.

A current source in series with a voltage source reduces to a current source with the same strength as is shown. i_1 can now be found by KCL and v_0 by KVL:



$$\text{KCL:} \qquad -i_1 - 10 - 15 = 0 \qquad \rightarrow \qquad i_1 = -25 \text{ A}$$

$$\text{KVL:} \qquad -100 + v_0 = 0 \qquad \rightarrow \qquad v_0 = 100 \text{ V}$$

Note that if we had reduced the two sources in parallel in the original circuit, we would have reached a circuit which was trivial and not helpful in finding v_0 and i_1 (try it!). So, circuit reduction should be used judiciously. Example below shows a circuit that can be solved without any circuit reduction and, in fact, circuit reduction makes the solution more difficult.

Example: In the circuit above, find v_0 and v_2.

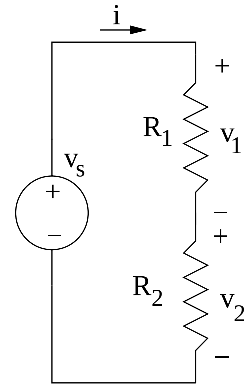
Both v_0 and v_2 an be found by KVL:

$$\text{KVL:} \qquad -100 + v_0 = 0 \qquad \rightarrow \qquad v_0 = 100 \text{ V}$$

$$\text{KVL:} \qquad -100 - 40 - v_2 = 0 \qquad \rightarrow \qquad v_2 = -140 \text{ V}$$

Some Practical Resistive Circuits

Voltage Divider:



The two resistors can be replaced by an equivalent resistor, $R_{eq} = R_1 + R_2$. Thus:

$$v_s = R_{eq} i \quad \to \quad i = \frac{v_s}{R_{eq}}$$

$$v_1 = iR_1 = \frac{R_1}{R_{eq}} v_s$$

$$v_2 = iR_2 = \frac{R_2}{R_{eq}} v_s$$

Also, $\dfrac{v_1}{v_2} = \dfrac{R_1}{R_2}$

This circuit is called a voltage divider as the two resistors divide the voltage of the IVS between them proportional to their values. This circuit can be extended by adding more resistors to the circuit and get more reference voltages.

This circuit is used extensively in electronic circuits. The basic reason is that power supplies are bulky and/or expensive. Typically, one power supply with one voltage is provided. On the other hand, more than one voltage may be needed for the circuit to operate properly.

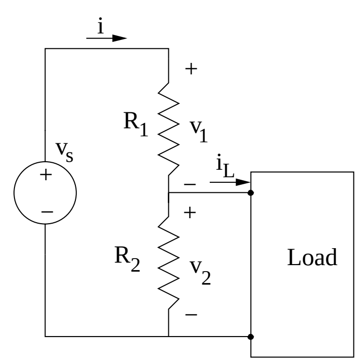
Example: A battery operated radio has a 9 V battery. Part of radio circuits require a 6 V supply. Design a voltage divider circuit to supply 6 V voltage to these circuits.

The desired circuit is the voltage divider circuit above with $v_s = 9$ V and $v_2 = 6$ V. Then,

$$v_2 = \frac{R_2}{R_{eq}} v_s \quad \to \quad 6 = \frac{R_2}{R_1 + R_2} 9 \quad \to \quad \frac{R_2}{R_1 + R_2} = \frac{6}{9}$$

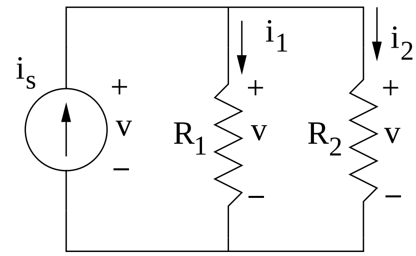
This is one equation in two unknowns and one is free to choose one parameter. For example, choosing $R_1 = 1$ kΩ we get $R_2 = 2$ kΩ.

Voltage dividers are affected by the load current drawn from them (see figure). The voltage divider formula can only be used for the circuit shown on the right if $i_l \ll i$ (prove it!). We will discuss the impact of the load on voltage dividers when we discuss real sources.



Current Divider:

The two resistors can be replaced by an equivalent resistor, $1/R_{eq} = 1/R_1 + 1/R_2$. Thus:



$$v = R_{eq} i$$

$$v = i_1 R_1 \quad \rightarrow \quad i_1 = \frac{v}{R_1} = \frac{R_{eq}}{R_1} i_s$$

$$v = i_2 R_2 \quad \rightarrow \quad i_2 = \frac{v}{R_2} = \frac{R_{eq}}{R_2} i_s$$

Also, $\quad \dfrac{i_1}{i_2} = \dfrac{R_2}{R_1}$

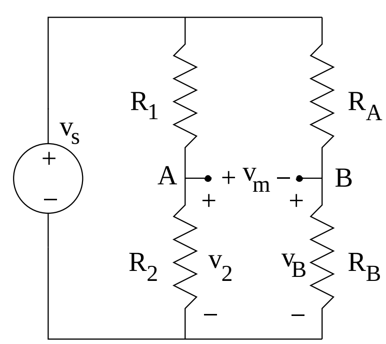
This circuit is called a current divider as the two resistors divide the current of the ICS between them (inversely proportional to their values). This circuit can be extended by adding more resistors to the circuit and get more reference currents.

Wheatstone Bridge

A typical Ohm-meter measures the resistance of a resistor by using the Ohm's Law. It applies a known voltage of v_s across the resistor, measures the current flowing through the resistor, and its dial are set to convert the measured value of current into the value of resistance by using $R = v_s/i_{measured}$. (This is why one cannot measure the value of a resistor while it is attached in a circuit, Ohm-meter works only if the resistor is not attached to anything but the meter.)

A typical digital multi-meter measure resistance within an accuracy of about 1%. In some cases, higher accuracy is needed. Resistor bridges are used for this purpose and they are made of two voltage divider circuits put in parallel with each other. The bridges operate based on the fact that while it is difficult to measure the difference between 1 and 1.01 V or 1 and 1.01 A distinctly (they are only 1% apart and within the accuracy of meter), it is easy to measure 0.01 V.

A most widely used bridge is the Wheatstone bridge shown. It consists of two voltage divider circuits with a voltmeter measuring the voltage between points A and B (denoted by v_m). We note from voltage divider formulas:



$$v_2 = \frac{R_2}{R_1 + R_2} v_s$$

$$v_B = \frac{R_B}{R_A + R_B} v_s$$

$$\text{KVL:} \qquad -v_2 + v_m + v_B = 0 \qquad \rightarrow \qquad v_m = v_2 - v_B = v_s\left(\frac{R_2}{R_1+R_2} - \frac{R_B}{R_A+R_B}\right)$$

A Wheatstone bridge is used in two modes. To measure the value of a resistor accurately, and to monitor to change in a resistance accurately.

Measuring resistance accurately: Suppose the unknown resistance is R_B. Two known and accurate resistors R_1 and R_A are chosen. An accurate but variable resistor is used for R_2. Typically, a resistor box or a decade box is used. The dials on the box switch some accurate resistor in parallel or in series to get the desired resistor value accurately. The bridge is setup and is powered up. The variable resistor R_2 is varied until the meter read zero voltage for v_m. Then:

$$v_m = v_s\left(\frac{R_2}{R_1+R_2} - \frac{R_B}{R_A+R_B}\right) = 0$$

$$\frac{R_2}{R_1+R_2} = \frac{R_B}{R_A+R_B} \qquad \rightarrow \qquad \frac{R_1+R_2}{R_2} = \frac{R_A+R_B}{R_B}$$

$$\frac{R_1}{R_2} = \frac{R_A}{R_B}$$

$$R_B = R_A\,\frac{R_2}{R_1}$$

Note that we do not need to know value of v_s to find R_B. This way, R_B is measure within the accuracy of resistors,R_1, R_2, and R_A.

Measuring resistance changes accurately: In certain sensors, the resistance of the sensor changes proportional to external forces or conditions. For example, a strain gauge measures the elongation (strain) of a solid material caused by applied forces (stress). A typical strain gauge consists of a thin film of conducting material deposited on an insulating substrate and bonded to a test member. When the test article is under stress, its dimension changes (*e.g.,* its length L changes to $L + \Delta L$. The resistance of the strain gauge (the conducting film) change according to

$$\Delta R = 2R_G\,\frac{\Delta L}{L}$$

where R_G is the resistance of gauge when no stress is applied, ΔR is the change in the gauge resistance, and factor of 2 comes from the fact that as the material is elongated its cross section is reduced. Usually, we like to measure strain values, $\Delta L/L$, that can be as small as 10^{-4} (which means the changes in gauge resistance is of the same order and cannot be measured by a simple ohm-meter).

Wheatstone bridge is used to measure ΔR in the following configuration. The strain gauge, R_G is put in place of R_2. R_1 is replaced by a similar strain gauge which is under no stress and is used as a reference. The resistances R_A and R_B are chosen such that bridge is balanced ($v_m = 0$) when no stress is applied. Following the equations for a balanced bridge, we get:

$$\frac{R_1}{R_1 + R_G} = \frac{R_B}{R_A + R_B}$$

Typically, $R_1 = R_G$ which implies $R_A = R_B$.

When stress is applied to the system, the strain gauge resistance changes to $R_2 = R_G + \Delta R$ and a voltage v_m appears on the bridge. Then,

$$v_m = v_s \left(\frac{R_G + \Delta R}{R_1 + R_G + \Delta R} - \frac{R_B}{R_A + R_B} \right)$$

Using $R_1 = R_G$ and $R_A = R_B$, we get:

$$v_m = v_s \left(\frac{R_G + \Delta R}{2R_G + \Delta R} - \frac{1}{2} \right)$$
$$v_m = v_s \, \frac{2R_G + 2\Delta R - 2R_G - \Delta R}{2(2R_G + \Delta R)}$$
$$v_m = v_s \, \frac{\Delta R}{2(2R_G + \Delta R)}$$
$$v_m = v_s \, \frac{\Delta R}{4R_G}$$

where in the last equation, we have ignored ΔR in the denominator since $\Delta R << R_G$. Using the relationship between ΔR and strain (ΔL), we get

$$\frac{\Delta L}{L} = \frac{\Delta R}{2R_G} = 2\,\frac{v_m}{v_s}$$

Note that in this application an accurate voltage source, v_s, is needed.