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Parametric hypoplasticity as continuum model for granular media: from Stokesium to Mohr-Coulombium and beyond

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Abstract Fluid-particle systems, in which internal forces arise only from viscosity or intergranular friction, represent an important special case of strictly dissipative materials defined by a history-dependent 4th-rank viscosity tensor. In a recently proposed simplification, this history dependence is represented by a symmetric 2nd-rank fabric tensor with evolution determined by a given homogeneous deformation. That work suggests an essential physical link between idealized suspensions ("Stokesium") and granular media ("Mohr-Coulombium") along with possible models for the visco-plasticity of fluid-saturated and dry granular media. The present paper deals with the elastoplasticity of dilatant non-cohesive granular media composed of nearly rigid, frictional particles. Based on the underlying physics and past modeling by others, a continuum model based on *parametric hypoplasticity* is proposed, which involves a set of rate-independent ODEs in the state-space of stress, void ratio and fabric. As with the standard theory of hypoplasticity, the present model does not rely on plastic potentials but, in contrast to that theory, it is based explicitly on positive-definite elastic and plastic moduli. The present model allows for elastic loading or unloading within a dissipative yield surface and also provides a systematic treatment of Reynolds dilatancy as a kinematic constraint. Some explicit forms are proposed and comparisons are made to previous hypoplastic models of granular media.

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1 Introduction

The strictly (or "purely") dissipative material [3,5] provides an eminently plausible constitutive framework for the quasi-static mechanics of wet or dry granular media dominated by intergranular friction and viscosity. If such a medium is free of internal kinematic constraints, the local Cauchy stress $\mathbf{T}(t)$ is given in terms of local deformation rate $\mathbf{D}(t)$ by the *pseudolinear* form:

$$\mathbf{T}(t) = \boldsymbol{\eta} : \mathbf{D}(t) \;\; (i.e.\;\; T_{ij}(t) = \eta_{ijkl} D^{kl}(t)), \tag{1}$$

where $\boldsymbol{\eta}$ denotes a positive-definite *viscosity* depending generally on $\mathbf{D}(t)$ and on local deformation history for past times $t' < t$, which for the present we denote by $\mathfrak{h} = \mathfrak{h}(t)$ but later specify more precisely.

Here, as in the following, we denote second-rank tensors by bold uppercase Roman and Greek symbols, $\mathbf{A}, \mathbf{B}, \ldots$, and fourth-rank tensors, regarded as linear operators on second-rank tensors, by bold lowercase Greek, $\boldsymbol{\alpha}, \boldsymbol{\beta}, \ldots$. Moreover, the respective idemfactors are denoted by $\mathbf{1}$, with components δ_{ij}, and by $\boldsymbol{\delta}$, with components $\delta_{ijkl} = \delta_{ik}\delta_{jl}$ (or an appropriately symmetrized form), and we employ standard tensor notation and summation convention for components on arbitrary curvilinear coordinates, with colons denoting ordered pairwise contraction on the trailing indices of prefactors with leading indices of postfactors. Primes denote deviators:

$$\mathbf{A}' := \mathbf{A} - \frac{1}{3}\mathbf{1}(\mathbf{1}:\mathbf{A}), \ldots \;\; \boldsymbol{\alpha}' := \boldsymbol{\alpha} - \frac{1}{3}\mathbf{1}(\mathbf{1}:\boldsymbol{\alpha}), \ldots \tag{2}$$

and superposed carat denotes the *versor* (or director) of real second rank tensors:

$$\widehat{\mathbf{A}} := \frac{\mathbf{A}}{|\mathbf{A}|}, \quad \text{where } |\mathbf{A}| := (\mathbf{A}:\mathbf{A}^{\mathrm{T}})^{1/2} \equiv A_{ij}A^{ij}, \tag{3}$$

while superscripts T and * denotes respective transpose or tensorial duals, with $A^{\mathrm{T}}_{ij} := A_{ji}$ and $\alpha^{*}_{ijkl} := \alpha_{klij}$, etc. As the vector-space generalization of the signum function, $\mathbf{B} = \widehat{\mathbf{A}}$ is defined for vanishing $|\mathbf{A}|$ as set-valued, with equality "=" to be interpreted as *inclusion* "$\in$".

The relation (1) provides a theoretical framework for the rheology of idealized rigid-particle suspensions in viscous fluids [5], representing the titular Stokesium. A general model of rigid plasticity is obtained from (1) on replacing $\boldsymbol{\eta}$ by $\boldsymbol{\mu}_{\mathrm{P}}/|\mathbf{D}|$, to give:

$$\mathbf{T}(t) = \boldsymbol{\mu}_{\mathrm{P}} : \widehat{\mathbf{D}}(t), \tag{4}$$

where $\boldsymbol{\mu}_{\mathrm{P}} = \boldsymbol{\mu}_{\mathrm{P}}(\mathfrak{h}, \widehat{\mathbf{D}})$ defines a positive-definite plastic modulus, and $\mathbf{D}$ is to be identified with plastic deformation rate, say, $\mathbf{D}_{\mathrm{P}}$. Here as in the following, subscripts P refer to quantities associated with plastic deformation.

As will become evident below, (4) provides both yield locus and flow rule [3]. With the proper dependence of $\boldsymbol{\mu}_{\mathrm{P}}$ on confining pressure, (4) provides a general theoretical framework for idealized assemblies of rigid frictional particles, the titular Mohr-Coulombium. The purpose of this article is to set down a more complete definition of the latter, together with a plausible extension to the elastoplasticity of granular media.

Here and below, we denote various 4th-rank elastoplastic moduli by the symbol $\boldsymbol{\mu}$ and the corresponding compliances by $\boldsymbol{\kappa} = \boldsymbol{\mu}^{-1}$, such that, whenever these represent invertible linear transformations on the space of second-rank tensors,

$$\boldsymbol{\kappa} : \boldsymbol{\mu} = \boldsymbol{\delta}, \quad (i.e.\ \kappa_{ijkl}\, \mu^{kl}_{\ \ \cdot\cdot mn} = \delta_{ijmn}), \tag{5}$$

Since $\boldsymbol{\mu}_{\mathrm{P}}$ is positive definite, hence invertible, (4) can be written in the inverse form as the *flow rule*:

$$\widehat{\mathbf{D}} = \boldsymbol{\kappa}_{\mathrm{P}} : \mathbf{T}, \tag{6}$$

where, in keeping with the usual plasticity modeling, $\boldsymbol{\kappa}_{\mathrm{P}} = \boldsymbol{\kappa}_{\mathrm{P}}(\mathfrak{h}, \mathbf{T})$. A connection is given in the Appendix to plastic potentials, which are not essential to the present theory.

Since $|\widehat{\mathbf{D}}| = 1$, a yield surface follows immediately from (6) as [3]:

$$Y := \|\mathbf{T}\|^2_{\zeta} := \mathbf{T} : \boldsymbol{\zeta} : \mathbf{T} = 1, \quad \text{where} \quad \boldsymbol{\zeta} = \boldsymbol{\kappa}^{*}_{\mathrm{P}} : \boldsymbol{\kappa}_{\mathrm{P}}, \tag{7}$$

with $\| \; \|_{\zeta}$ representing dissipation norm. Then, the condition $Y < 1$ on this norm can be taken to define rigid states lying within the yield surface.

1.1 Rigid grains (Mohr-Coulombium)

We now consider a form of the above theory appropriate to assemblies of noncohesive, perfectly rigid grains undergoing quasi-static deformations. Since such a medium is devoid of inherent time scale, a rate-independent form like (4) is immediately applicable. As additional important properties, we expect such a medium to exhibit (1) Reynolds dilatancy and (2) no characteristic internal stress.

Following Reynolds [15], dilatancy is treated as a (nonholonomic) kinematic constraint, expressed here by a coefficient of dilatancy α [6]:

$$\mathrm{tr}(\mathbf{D}) \equiv \mathbf{1} : \mathbf{D} = \alpha |\mathbf{D}'|, \quad \text{with} \quad \alpha = \alpha(\mathfrak{h}, \tilde{\mathbf{D}}),$$
$$\tilde{\mathbf{D}} = \widehat{\mathbf{D}}' \equiv \frac{\mathbf{D}'}{|\mathbf{D}'|}, \tag{8}$$

This represents a general cone in $\mathbf{D}$-space, with polar axis defined by the isotropic direction $\widehat{\mathbf{1}}$ and azimuthal hyperplanes by $\tilde{\mathbf{D}}$. As a matter to be discussed below, note that the special case of homogeneous isotropic compression with $\mathbf{D}' = \mathbf{0}$ is not allowed by (8). Furthermore, owing to the dilatancy constraint, $\mathrm{tr}(\mathbf{D})$ can no longer be derived from (6), and we must replace $\widehat{\mathbf{D}}$, $\boldsymbol{\kappa}_{\mathrm{P}}$ by their deviatoric forms in (6) and (7). In effect, this renders (6) non-invertible, and, by a slight extension of a well-known principle for holonomic material constraints [17], $\mathbf{T}$ is determined only up to to an additive, purely reactive stress $\mathbf{T}_{\mathrm{R}}$ satisfying $\mathbf{T}_{\mathrm{R}} : \mathbf{D} = 0$ for all kinematically admissible $\mathbf{D}$, as defined by (8). This reactive stress defines a second cone, with generators orthogonal to the cone (8), which represents the (Reynolds-Rowe) yield surface for a hypothetical frictionless assembly having the same α as the actual assembly [6].[1]

The above cones are conveniently expressed as dual pseudolinear forms:

$$\mathbf{D} : (\tilde{\mathbf{D}} - \tan\phi_{\mathrm{D}} \widehat{\mathbf{1}}) = 0 \;\&\; \mathbf{T}_{\mathrm{R}} : (\tilde{\mathbf{D}} + \cot\phi_{\mathrm{D}} \widehat{\mathbf{1}}) = 0,$$
$$\text{with} \;\; \phi_{\mathrm{D}} = \mathrm{arccot}\frac{\alpha}{\sqrt{3}}, \tag{9}$$

illustrated schematically in Fig. 1. With $\mathbf{1}$ representing polar axis, $\phi_{\mathrm{D}}(\mathfrak{h}, \tilde{\mathbf{D}})$ represents the polar angle of dilatancy (different from that employed by Roscoe)[6].

The reaction stress can be expressed generally as the projection [12, Eq. (5.13.4)]

$$\mathbf{T}_{\mathrm{R}} = \boldsymbol{\pi} : \boldsymbol{\Lambda}, \quad \text{where} \;\; \boldsymbol{\pi} := \boldsymbol{\delta} - \frac{(\tilde{\mathbf{D}} + \alpha\mathbf{1}/3) \otimes (\tilde{\mathbf{D}} + \alpha\mathbf{1}/3)}{(1 + \alpha^2/3)}, \tag{10}$$

with $\boldsymbol{\Lambda}$ denoting an arbitrary symmetric 2nd-rank tensor. However, since the dilatancy constraint removes but one degree of freedom, $\boldsymbol{\Lambda}$ can be treated as known up to a rheologically indeterminate scalar multiplier, with $\widehat{\boldsymbol{\Lambda}}$ given possibly by yet another constitutive equation.

We note further that any $\boldsymbol{\Lambda}$ not contained in the (hyper) plane spanned by $\tilde{\mathbf{D}}$ and $\mathbf{1}$ involves a gyration or "twist" of $\mathbf{T}_{\mathrm{R}}$ out of this plane [6]. Ruling out such an effect by choosing $\boldsymbol{\Lambda}$ to be a linear combination of $\tilde{\mathbf{D}}$ and $\mathbf{1}$, we find from

[1] It should be noted that certain quasi-static numerical simulations [6] yield a non-zero value of α for frictionless spheres, whereas others [14,16] give $\alpha = 0$, somewhat at odds with the ideas of Reynolds [15].

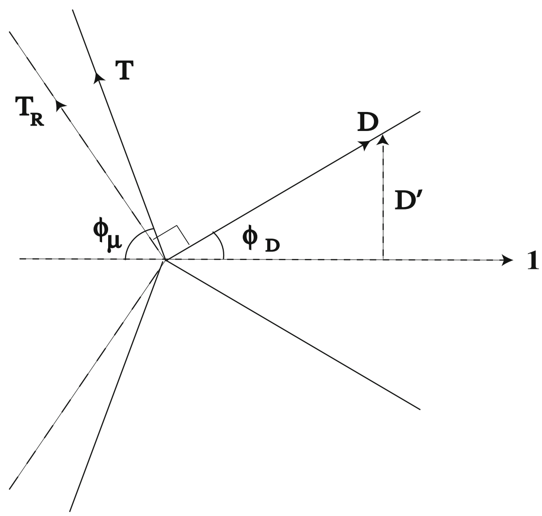

Fig. 1 Schematic of dilatancy cone and orthogonal cone of reaction stress $\mathbf{T}_R$, in a meridional hyperplane $\tilde{\mathbf{D}} = $ const. of $\mathbf{D}$-space

(10) that $\mathbf{T}_\mathrm{R}$ is proportional to $\alpha\tilde{\mathbf{D}} - \mathbf{1}$. Hence, the inverse of (6) can be written

$$\mathbf{T} = \boldsymbol{\mu}'_\mathrm{P} : \tilde{\mathbf{D}} + \alpha\nu_\mathrm{P}\mathbf{1} + p(\alpha\tilde{\mathbf{D}} - \mathbf{1}), \quad \text{where } \mathbf{1} : \boldsymbol{\mu}'_\mathrm{P} = \mathbf{0}, \quad (11)$$

where $p = -\mathrm{tr}(\mathbf{T}_\mathrm{R})/3$ represents the reaction to the dilatancy constraint, assuming its well-known role in the limiting case of an incompressible material $\alpha \equiv 0$, and the scalar ν_P describes an isotropic friction in compression. The positivity of

$$\mathbf{T} : \tilde{\mathbf{D}} \equiv \tilde{\mathbf{D}} : \boldsymbol{\mu}'_\mathrm{P} : \tilde{\mathbf{D}} + \nu_\mathrm{P}\alpha^2, \quad (12)$$

requires positivity of $\boldsymbol{\mu}'_\mathrm{P}$, ν_P.

The absence of cohesion or other characteristic stress dictates that stresses appear in constitutive equation only as stress ratios and that the confining pressure be positive. This defines yet another cone, whose form is rendered definite by choosing the coefficients in (11) proportional to $p \geq 0$. In this case, (11) can be rewritten as

$$\tilde{\mathbf{T}} := \mathbf{T}/p = \boldsymbol{\mu}'_\mathrm{C} : \tilde{\mathbf{D}} + \alpha\tilde{\mathbf{D}} + (\alpha\nu_\mathrm{C} - 1)\mathbf{1}, \quad (13)$$

where $\boldsymbol{\mu}'_\mathrm{C} = \boldsymbol{\mu}'_\mathrm{C}/p$ and $\nu_\mathrm{C} = \nu_\mathrm{P}/p$ represent non-dimensional (Coulomb) friction coefficients, and (13) serves to define the cone

$$\left.\begin{aligned} &|\mathbf{T}'| = \sqrt{3}\, p \tan\phi_\mu, \quad \text{or } \mathbf{T} : \left(\tilde{\mathbf{T}} + \tan\phi_\mu \hat{\mathbf{1}}\right) = 0, \\ &\phi_\mu = \arctan\left\{\left(\tilde{\mathbf{D}} : \boldsymbol{\mu}'^{*}_\mathrm{C} : \boldsymbol{\mu}'_\mathrm{C} : \tilde{\mathbf{D}} + \alpha^2/3\right)^{1/2}\right\} \end{aligned}\right\} \quad (14)$$

ϕ_μ defines an apparent angle of internal friction, and the corresponding yield surface is illustrated schematically in Fig. 1. It generally does not have an isotropic or "circular" (Drucker-Prager) form [6].

This completes our description of rigid plasticity, and we now consider the modifications necessary to account for slightly elastic particles.

2 Elastoplasticity and hypoplasticity

To account for elastic effects, we adopt the usual distinction between elastic and plastic deformation rates, with plastic rate $\mathbf{D}_\mathrm{P}$ given by (6) and elastic rate by a tangential elastic compliance, say $\boldsymbol{\kappa}_\mathrm{E}(\mathfrak{h}, \mathbf{T})$, and

$$\mathbf{D}_\mathrm{E} = \boldsymbol{\kappa}_\mathrm{E} : \overset{\circ}{\mathbf{T}}, \quad (15)$$

where $\overset{\circ}{\mathbf{T}}$ represents an objective (e.g. Jaumann) rate. Then, the standard incremental elastoplasticity based on additive rates gives

$$\mathbf{D}_\mathrm{E} + \mathbf{D}_\mathrm{P} = \boldsymbol{\kappa}_\mathrm{E} : \overset{\circ}{\mathbf{T}} + |\mathbf{D}'_\mathrm{P}| \left(\boldsymbol{\kappa}'_\mathrm{P} : \mathbf{T} + \frac{1}{3}\alpha\mathbf{1}\right) = \mathbf{D} \quad (16)$$

which can be rewritten in the usual hypoplastic form as

$$\left.\begin{aligned} &\overset{\circ}{\mathbf{T}} = \boldsymbol{\mu}_\mathrm{H} : \mathbf{D} - |\mathbf{D}|\mathbf{N}, \\ &\mathbf{N} := \frac{\vartheta}{\sqrt{1+\alpha^2/3}} \boldsymbol{\mu}_\mathrm{H} : \left(\boldsymbol{\kappa}'_\mathrm{P} : \mathbf{T} + \frac{1}{3}\alpha\mathbf{1}\right), \quad \vartheta := \frac{|\mathbf{D}_\mathrm{P}|}{|\mathbf{D}|}, \ \boldsymbol{\mu}_\mathrm{H} \equiv \boldsymbol{\mu}_\mathrm{E} \end{aligned}\right\} \quad (17)$$

However, the standard hypoplastic model [10] takes $\vartheta \equiv 1$, making no distinction between $\mathbf{D}_\mathrm{P}$ and $\mathbf{D}$, and does not rely on the existence of a yield surface[2]. Also, $\boldsymbol{\mu}_\mathrm{H}$ and $\mathbf{N}$ are taken to be isotropic tensor functions of $\mathbf{T}(t)$, and $\boldsymbol{\mu}_\mathrm{H}$ is not necessarily symmetric or positive definite.[3]

After consideration some salient features of (17), we show below how it can be considered part of a more general hypoplasticity.

The rate-independent form of (17) allows one to eliminate time t in favor of accumulated plastic strain:

$$t_\mathrm{P} = \int_0^t |\mathbf{D}_\mathrm{P}|(t')dt' = \int_0^t \vartheta|\mathbf{D}|(t')dt' \quad (18)$$

However, in the elastoplastic model this time-strain map requires a further constitutive relation ("inelastic clock") for ϑ in (17) to describe the evolution of t_P, e.g. a relation of the (Hill-Rice) form:

$$\dot{t}_\mathrm{P} = \vartheta(\mathfrak{h}, \mathbf{T})|\mathbf{D}| \quad (19)$$

where $\vartheta = 0$ in elastic states. Consistent with (7), one possibility is

$$\vartheta = H(Y - 1), \quad (20)$$

[2] whose precise definition is questioned even by eminent exponents of the classical theory [8].

[3] Gurtin [7] concludes that a plasticity with yield locus follows from the *hypoelastic* form [10,17] of (17), $\mathbf{N} \equiv \mathbf{0}$, but it is not clear how the implied history dependence can be achieved if $\boldsymbol{\mu}_\mathrm{H}$ depends only on present stress.

where H denotes the Heaviside step with $H(0) := H(0+)=1$. Hence, given evolution equations for $\mathfrak{h}$, one has complete evolution equations for stress.

Interpreting the elastic compliance in (15) in terms of a complementary strain energy

$$\boldsymbol{\kappa}_{\mathrm{E}} = \partial_{\mathbf{T}} \otimes \partial_{\mathbf{T}} \varphi_{\mathrm{E}}, \tag{21}$$

we find that

$$\left.\begin{aligned} &\int \mathbf{T} : \mathbf{D}_{\mathrm{E}} dt = \int \mathbf{T} : \boldsymbol{\kappa}_{\mathrm{E}} : \overset{\circ}{\mathbf{T}}\, dt = \int \partial_{\mathbf{T}} \psi_{\mathrm{E}} : d\mathbf{T}, \\ &\text{where} \\ &\psi_{\mathrm{E}} = \mathbf{T} : \partial_{\mathbf{T}} \varphi_{\mathrm{E}} - \varphi_{\mathrm{E}} \end{aligned}\right\} \tag{22}$$

Hence, for the dilatant plasticity model defined by (11), the stress work on any cycle is given by

$$\begin{aligned} \oint \mathbf{T} : \mathbf{D} dt \; = \; & \oint \partial_{\mathbf{T}} \psi_{\mathrm{E}} : d\mathbf{T} \\ & + \oint (\tilde{\mathbf{D}}_{\mathrm{P}} : \boldsymbol{\kappa}'_{\mathrm{P}} : \tilde{\mathbf{D}}_{\mathrm{P}} + \nu_{\mathrm{P}} \alpha^2) |\mathbf{D}'_{\mathrm{P}}| dt, \end{aligned} \tag{23}$$

which is obviously positive for cycles on which ψ_{E} can be regarded as a function of $\mathbf{T}$ alone. This constitutes a weak version of Il'yushin's postulate [11] and distinguishes the present model from the most general form of (17).

2.1 Parametric hypoplasticity

With a view to granular media, Wu and Bauer [1,18,19] have proposed variants of the standard hypoplastic model, the more recent of which [1] allow $\boldsymbol{\mu}_{\mathrm{H}}$ and $\mathbf{N}$ to depend on void ratio e. They adopt the well-known relation for incompressible grains

$$\dot{e} = (1 + e)\mathrm{tr}(\mathbf{D}), \tag{24}$$

with no kinematic restriction on $\mathrm{tr}(\mathbf{D})$. This provides a phenomenological description of isotropic compaction, compatible with critical-state soil mechanics, at pressures comparable to a characteristic pressure (given by their "hardness" h_s). This same parameter allows for a cap on the yield locus defined by vanishing of stress rate in (16, 17). The Wu–Bauer model represents dynamics in state-space $\mathcal{X} = \{\mathbf{T}, e\}$ governed by an ODE, homogeneous degree-one in control variable $\mathbf{D}$:

$$\overset{\circ}{\mathcal{X}} = \mathcal{H}(\mathcal{X}, \mathbf{D}), \text{ with } \mathcal{H}(\mathcal{X}, \lambda\mathbf{D}) = \lambda\mathcal{H}(\mathcal{X}, \mathbf{D}), \quad \lambda \geq 0, \tag{25}$$

as anticipated in early works on hypoplasticity [9]. The underlying model, which we denote here as *parametric hypoplasticity*, also provides a model of rate-independent hysteresis in numerous other settings.

Provided $\boldsymbol{\kappa}_{\mathrm{E}}, \boldsymbol{\kappa}'_{\mathrm{P}}, \alpha$ depend only on $\mathcal{X}$, the elastoplastic model (17) can also be represented by parametric hypoplasticity, with $\mathcal{X} := \{\mathbf{T}, t_{\mathrm{P}}, e\}$ and

$$\dot{e} = (1 + e)\mathrm{tr}(\mathbf{D}_{\mathrm{P}}) = \frac{(1 + e)\alpha\vartheta}{\sqrt{1 + \alpha^2/3}} |\mathbf{D}|, \tag{26}$$

which follows from the standard multiplicative decomposition of deformation gradient [11] and reduces to (24) for $\mathbf{D} \to \mathbf{D}_{\mathrm{P}}, \alpha \to \infty$. Conversely, an incompressible fluid-saturated granular medium is represented by $\alpha = 0$.

The ODE (25) is now represented by the set (17), (19, 20) and (26), with (19) and (26) providing an ODE for de/dt_{P}. Thus, without any appeal whatsoever to plastic potentials, we obtain a proper dilatant variant of standard elastoplasticity with plastic strain t_{P}, given by (19, 20), as the sole descriptor of deformation history $\mathfrak{h}$.

Note that (26) allows for isotropic compression, $\mathbf{D}' = \mathbf{D}'_{\mathrm{P}} + \mathbf{D}'_{\mathrm{E}} = \mathbf{0}$, construed to arise from the deformation of elastic grains. In this regard, we recall that the packing density of rigid spheroids changes rapidly in the near-sphere limit [2], suggesting that small elastic deformations could have large effects on granular dilatancy. In the case of stiff particles, with $\alpha \gg 1$, we have

$$\mathrm{tr}(\mathbf{D}) = \mathbf{1} : \boldsymbol{\kappa}_{\mathrm{E}} : \overset{\circ}{\mathbf{T}} + \alpha |\boldsymbol{\kappa}'_{\mathrm{E}} : \overset{\circ}{\mathbf{T}}\,| = \alpha |\boldsymbol{\kappa}'_{\mathrm{E}} : \overset{\circ}{\mathbf{T}}\,| \{1 + O(\alpha^{-2}\})\}, \tag{27}$$

This relation is compatible with "stiff" elastoplasticity [13, pp. 301 ff.], represented here by

$$\epsilon = \frac{||\boldsymbol{\kappa}_{\mathrm{E}}||}{||\boldsymbol{\kappa}_{\mathrm{P}}||} = \frac{p||\boldsymbol{\kappa}_{\mathrm{E}}||}{||\boldsymbol{\kappa}_{\mathrm{C}}||} << 1, \quad \text{with } \alpha \propto \epsilon^{-1} \tag{28}$$

In this limit, (16) reduces to a stiff ODE, with $\mathbf{D} \approx \mathbf{D}_{\mathrm{P}}$, except on small elastic strain scales $\Delta t_{\mathrm{P}} = O(\epsilon)$ near points of elastic loading or unloading. This model appears particularly appropriate to relatively stiff geomaterials such as sand, where the continuum-level elasticity can be associated with Hertzian-type contact between nearly rigid grains.

As suggested by previous models [19], the compatibility of (27, 28) with critical-state soil mechanics would no doubt require a more complicated dependence of the dilatancy coefficient α on confining pressure.

2.2 Anisotropy and fabric

In the models considered above, the 4th-rank moduli and compliances denoted by $\boldsymbol{\mu}, \boldsymbol{\kappa}$ etc. are assumed given as either functions of current stress $\mathbf{T}$ or current deformation rate $\mathbf{D}$, together with scalar quantities such as void ratio and accumulated plastic strain. If the models are assumed to represent isotropic materials, then the moduli and compliances can be represented as isotropic polynomials in $\mathbf{T}$ or $\mathbf{D}$, as proposed in prior literature on the subject [4,10].

In the context of hypoplasticity, the early work of Kolymbas [9] recognizes the need for additional internal variables to describe strain-induced anisotropy, in particular a 2nd-rank tensor, which can be identified here with the granular fabric tensor $\mathbf{A}$ [5,13]. The evolution of $\mathbf{A}$, governed by hypoplastic ODEs, defines the dependence on deformation history $\mathfrak{h}$. A more general set of internal variables: $\mathcal{A} = \{\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n\}$, $\mathbf{A}_1 \equiv \mathbf{A}$ is defined by the hypoplastic system:

$$\left.\begin{array}{l} \overset{\circ}{\mathbf{A}}_k = |\mathbf{D}'_{\mathrm{P}}|\mathbf{A}_{k+1}, \quad \text{for } k = 1, \ldots, n-1, \\ \overset{\circ}{\mathbf{A}}_n = \mathbf{H}(\mathcal{X}, \mathbf{D}'_{\mathrm{P}}), \quad \mathcal{X} = \{\mathbf{T}, e, \mathcal{A}\} \end{array}\right\} \tag{29}$$

where $\mathbf{H}$ represents a hypoplastic form like that in (17) and $\circ$ denotes the objective rate introduced above.

Recent work on dense fluid-particle suspensions [5] indicates that the lowest-order model, $n = 1$, with $\mathrm{tr}(\mathbf{A}) = 0$, suffices to describe induced anisotropy in simple shear. A similar simplification for dry granular media would give rise to the form (25) proposed above, as represented by (17, 26), and the last two equations of (29), where $\mathcal{A} \equiv \mathbf{A}$, and where $\mathbf{H}$, $\boldsymbol{\kappa}_{\mathrm{E}}$, $\boldsymbol{\kappa}'_{\mathrm{C}}$, ν_{C}, α are given as isotropic functions of $\mathcal{X}$. In this case, $\overset{\circ}{\mathcal{X}}$ can be expressed in terms of isotropic tensor polynomials in $\mathbf{T}$, $\mathbf{A}$ [5,9,10,18,19]. We further recall that Wu et al. [18,19] obtain good empirical fits of similar models to triaxial and cyclic shear tests on sand, even without explicit dependence on $\mathbf{A}$. This suggests that one might obtain an adequate model for more complex deformations with a low-order expansion in $\mathbf{A}$ of the type employed for suspensions [5].

3 Conclusions

A theoretical framework has been proposed for the elastoplasticity of non-cohesive, stiff granular media based on the notion of parametric hypoplasticity. The theory incorporates Reynolds dilatancy as kinematic constraint and also allows for strain-induced anisotropy based on the evolution of granular fabric. The success of empirical models based on simpler isotropic versions of hypoplasticity [18,19] make them plausible starting point for further refinement based on the present theory.

Appendix: plastic potentials

A sufficient (but not necessary) condition for the positivity of $\boldsymbol{\kappa}_{\mathrm{P}}$ is that it be given by the Hessian of a convex potential $\varphi_{\mathrm{P}} = \varphi_{\mathrm{P}}(\mathfrak{h}, \mathbf{T})$:

$$\boldsymbol{\kappa}_{\mathrm{P}} = \partial_{\mathbf{T}} \otimes \partial_{\mathbf{T}} \varphi_{\mathrm{P}} \ \ (i.e.\ \kappa_{ijkl} = \partial_{T^{ij}} \partial_{T^{kl}} \varphi), \tag{30}$$

whence one obtains a well-known flow rule:

$$\widehat{\mathbf{D}}_{\mathrm{P}} = \partial_{\mathbf{T}} \psi_{\mathrm{P}} \equiv \mathbf{T} : \partial_{\mathbf{T}} \otimes \partial_{\mathbf{T}} \varphi_{\mathrm{P}}, \quad \text{where} \ \ \psi_{\mathrm{P}} = \mathbf{T} : \partial_{\mathbf{T}} \varphi_{\mathrm{P}} - \varphi_{\mathrm{P}}, \tag{31}$$

ψ_{P}, φ_{P} representing a dissipative potential and its Legendre complement or dual, respectively. Then, (4) represents the inverse form:

$$\mathbf{T} = \widehat{\mathbf{D}} : \partial_{\widehat{\mathbf{D}}_{\mathrm{P}}} \otimes \partial_{\widehat{\mathbf{D}}_{\mathrm{P}}} \psi_{\mathrm{P}} \equiv \partial_{\widehat{\mathbf{D}}_{\mathrm{P}}} \varphi_{\mathrm{P}}, \quad \text{with} \ \ \boldsymbol{\mu}_{\mathrm{P}} = \partial_{\widehat{\mathbf{D}}_{\mathrm{P}}} \otimes \partial_{\widehat{\mathbf{D}}_{\mathrm{P}}} \psi_{\mathrm{P}} \tag{32}$$

The standard Legendre transformation treats ψ_{P} as function of $\mathbf{T}$ and φ_{P} as function of $\widehat{\mathbf{D}}_{\mathrm{P}}$, but invertibility allows for the interchange of roles.

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