SYSTEMS MODELING AND MODEL REDUCTION

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Contents

Glossary

Range space: A column vector x such that $x = Ay$ for some column vector y is said to be in the *range space* of matrix A, i.e., $x \in \mathcal{R}(A)$.

Null space: A column vector x such that $Ax = 0$ is said to be in the *right null space* of matrix A, i.e., $y \in \mathcal{N}_R(A)$. A row vector x such that $xA = 0$ is said to be in the *left null space* of A, i.e., $y \in \mathcal{N}_L(A)$. In this text the symbol $(^\perp)$ is used to denote a basis to $\mathcal{N}_L(A)$. That is, $x \neq 0 \in \mathcal{N}_L(A)$ if and only if $\exists y \neq 0 : x = yA^\perp$.

SISO system: Single-input-single-output system.

MIMO system: Multiple-input-multiple-output system.

Singular values: Any real matrix $A \in \mathbb{R}^{m\times n}$ admits a decomposition in the form $U\Sigma V = A$ where $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ are unitary matrices ($U^TU = I$, $V^TV = I$) and $\Sigma \in \mathbb{R}^{m\times n}$ is a matrix with positive diagonal entries σ_i, $i = 1,\ldots,\min(m,n)$, and zeros elsewhere. The symbol $\bar{\sigma}$ denotes the maximum singular value, i.e., $\bar{\sigma} := \max_i \sigma_i$.

Positive definite matrix: A real, square and symmetric matrix $A \in \mathbb{R}^{n\times n}$ is said to be positive (negative) definite if the inequality (*quadratic form*) $x^TAx > 0$ ($x^TAx < 0$) holds for all vectors $x \in \mathbb{R}^n$, $x \neq 0$. Equivalently, A is positive (negative) definite if all eigenvalues of A are positive (negative). Notice that A has only real eigenvalues due to symmetry.

Summary

Modeling physical systems usually results in complex high-order dynamic models. It is often desirable to replace (approximate) these models by simpler models with reduced order. In this process it is important to *design* the reduced model so as to capture the *important* properties of the original high-order model. This chapter describes some procedures that are available for the model reduction of linear time-invariant systems.

1. Introduction

Throughout history, quantum leaps in technology have occurred when certain technical ingredients were sufficiently mature to launch a new discipline to integrate old ones. Relativity provided integrations of field theory and classical dynamics. Control science emerged from a maturation of the necessary mathematics, computer science, and a maturation of sensing/actuator technology. The intersection of fluid dynamics and structures created "aeroelasticity" fields of study. What will become of structural control? A revolution is before us at the interface of material science, system science, signal processing, and structural engineering. On the near horizon, structural systems will result from the integration of material science, system science and control science. The structural systems of the future will integrate the functions of material to simultaneously provide communication channels, sensing networks, actuating, thermal management, antenna functions, and power generation. Algorithms will utilize the information provided by such networks to predict life, detect failures, update models and control laws, and even update and relieve stresses in the structure. It might be said that physics paved the way for technology in the first half of the 20th century, and engineering paved the way in the second half. What will drive and enable new technology as we round the corner of the new millennium? Engineering has produced sophisticated component technologies without instructions how to put them together. We have multidisciplinary problems begging for an interdisciplinary theory. There is ample evidence that analytical and numerical tools for interdisciplinary research (loosely called "system design") will be the next major enabler of technology. This idea is more than a century old. Michael Faraday said "we should begin with the whole, then construct the parts". Folloing the lead of Faraday, we refer to system design as the task of determining the design requirements of the multiple components that make up the system, given only the requirements of the system.

Universities teach component technology, drawing narrow boundaries around subjects for tractability, or easy management. Of course, this training leads the industrial project managers and engineers to uncoordinated decisions about system design. For example, *after* we select what components to manufacture, we have good guidelines how to manufacture them. *After* we choose components we have good first principles how to model them. *After* we choose the distribution of sensors and information, and if all other components are already designed, then there are guidelines how to control the system using existing control theory. Synergism is a popular concept, but when is the whole less than the sum of the parts? In system design, the answer is usually. In the absence of a necessity theory, we overdesign the components, dealing with what is sufficient, rather than what is necessary. However, contrary to popular opinion, the best system is not necessarily constructed from the best parts. We can waste money on components that are not necessary, or by making a component too precise. There is often more to be gained by unifying two disciplines than by improving the technology of either discipline. Yet, the best system performance (defined to include cost) cannot be achieved simply by inserting the best component technologies.

The first challenge is to overcome language barriers, to describe what's new using old words. The words "system" and "design" are much overused words that must be explained carefully before any real communication can take place. There are a number of "system design" principles available. Over the last twenty years much has been written about "concurrent" engineering, and "system engineering management" principles. Control theory is also a component technology in the sense that all the other components are assumed to be first specified. We believe that the next grand challenge is to enlarge the existing tools of control design and structure design to embrace the much more general problem of system design.

The sequence in which the components of a system are designed and introduced into the system dynamics can be quite varied. Traditionally, control has been the last component to design. Indeed, the decision whether control design should occur first, last, or during the structure design is a recurring theme in this chapter. At some universities, "design" is not viewed as a scholarly activity, because a scientific method to do it is lacking.

How can one identify the performance-limiting technology in a design? Perhaps we could get better performance by improving the manufacturing precision of selected components. If so, which components? Maybe we need a better model of the system components. If so, which components? Or maybe we're limited by computational precision, or, maybe by the algorithm we are using for signal processing or control design. Or, maybe we need better sensors, or a different distribution of the sensor or actuator locations. Or, maybe, more money in any of the component technologies would be wasted, because we are already operating at the fundamental limits of physics. Oddly enough, a systems theory is not available that even allows one to answer these basic questions. We have no systematic process to decide where to spend money to enable a certain level of performance of the system.

1.1. The Critical Challenge : System Modeling

The point of this section is that good component models do not imply good models for system design purposes. Indeed, the critical challenge in finding a system design methodology is to find a system modeling methodology. Component models are usually considered good if the output error is zero, but for any model, *M* in Figure 1, there is always some input to excite unmodeled dynamics, making the open-loop error, e, large. Clearly, modeling decisions for a component ought to be influenced by inputs from neighboring components that dynamically interact with this component. For example, the basis functions chosen to model the displacement in a structure ought to be influenced by the inputs. Even though there exist natural basis functions from physics to match stated boundary conditions to make e small, these are not the most appropriate basis functions for other inputs, such as control. Recall also from Mullis & Roberts that the optimal basis for a physical component depends on the disturbances in the computational component.

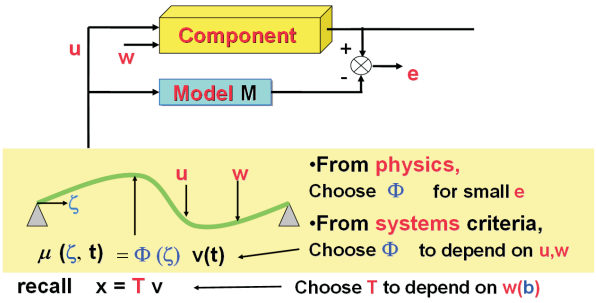


Figure 1: Component Modeling

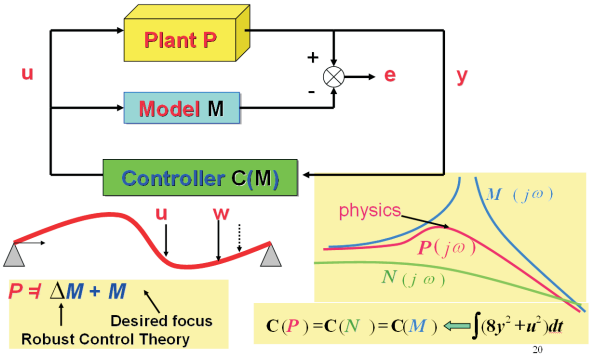


Figure 2: Is System Modeling Just Physics?

Modeling is not just Physics

Let's discuss 3 models of a plant, in Figure 2. *P* is the exact model from physics, a damped 2nd order system. M and N are erroneous models. In fact, both models have unbounded errors relative to the exact model, P. M is an undamped 2nd order system and N is a first order system. Now suppose the models are to be used for control design to minimize some specific objective function, shown in

the Figure 2. For this particular objective function, all 3 models, P,M,N, yield the same controller. Note that these models are arbitrarily far apart, by open loop criteria, but entirely equivalent by a closed loop criteria. Hence, the same controller can be optimal for many different models of the plant besides the exact model from physics. In fact, there might exist a better model for System Design than the Actual model from physics. The first order model would be simpler and yet yields the same controller. This point needs emphasis. Not only is the exact model inferior, assuming that it could be available, but searching for the exact model is more costly. Hence, modeling for system design is not just physics, physics, physics of the components, to make the open loop error smaller. Yet bounding this open loop error is the focus of robust control. These examples show that bounding the open loop error is neither necessary nor sufficient for a good control design. The conclusion is that "An Ounce of Model Improvement is worth a Ton of Robust Control". System Modeling is a discipline. It seems to require more than just Component Technology, and more than what each discipline (including controls) already knows.

2. What is Model Reduction?

The description of a physical dynamic system by a set of differential (or difference) equations is a very useful tool in science. These equations, refereed here as a mathematical *model*, can be obtained from basic physical principles or as a result of experiments. A measure of the "complexity" of the system model is the number of first order equations used to describe it. This number is often referred as the *order* of the model. Models with elevated order are able to describe very complex phenomena. Consequently, models with high order may be required in order to provide an accurate description of a dynamic system. For instance, models with an infinite number of differential equations often appears in several fields. To name one, the behavior of materials based on continuum physics is often described by partial differential equations or by an infinite number of ordinary differential equations.

If the capacity of a model to accurately describe a system seems to increase with the order of the model, in practice, models with low orders are required in many situations. In some cases, the amount of information contained in a complex model may obfuscate simple, insightful behaviors, which can be better captured and explored by a model with low order. In cases such as control design and filtering, where the design procedures might be computationally very demanding, limited computational resources certainly benefit from low order models. These examples justify the need to develop procedures that are able to approximate complex high order models by generating adequate *reduced order models*. As a result, some degree of detailing will be permanently lost in the reduced order model. The differences between the dynamics of the high order model and the obtained low order model (the *unmodeled* dynamics) can be often taken into account in the low order model as a *noise*, which can be handled using stochastic process methods. In any case, the model reduction procedures might be flexible enough to let the user indicate the essential behaviors that need to be captured for its application.

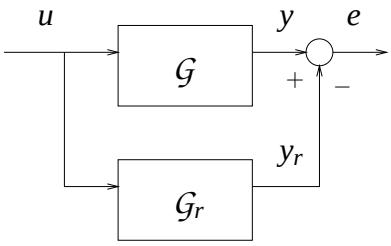
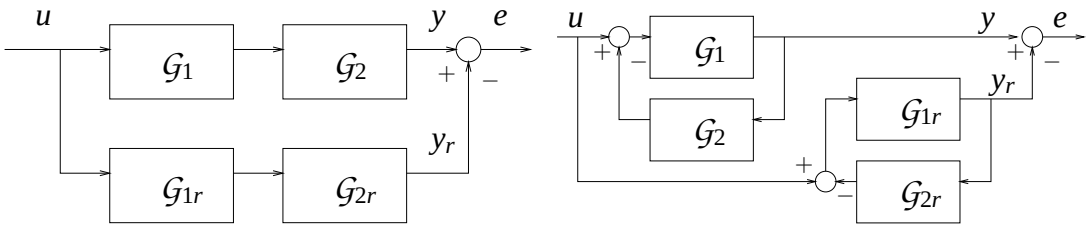


Figure 3: Single Component Model Reduction



(a) Series Connection

(b) Feedback Connection

Figure 4: Multi-Component Model Reduction

2.1. Single Component Model Reduction

Given a dynamic system model $\mathcal{G}$ of usually high order n, a model reduction method is a procedure that yields some approximate model $\mathcal{G}_r$ of order $n_r < n$. The quality of the approximation is usually evaluated by looking at the *model reduction error*, that is, the signal obtained as the difference between the outputs of the original system and the outputs of the reduced order model driven by the same input signal. That arrangement is depicted in Figure 3. With respect to this figure, the single component model reduction problem can then be loosely stated as follows:

> Given a system $\mathcal{G}$, chose a procedure that yields a reduced order model $\mathcal{G}_r$ so that the model reduction error is small.

2.2. Multi-Component Model Reduction

Quite often model reduction procedures might be applied to one component of a system while the other component to which it is connected remain the same. Two examples of such occasion are given. Figure 4(a) characterizes two components of a system interconnected in series. As an illustration, if this system represents a spacecraft, component $\mathcal{G}_1$ might represent a solar panel while component $\mathcal{G}_2$ could be the body of the spacecraft. The solar array and the body of the spacecraft might be manufactured by different companies and it would be useful to know how to write contracts so that the two companies can manufacture products with *component* dynamic properties such that the connections of the two components produce *system* dynamic properties that meet stringent specifications. The contracts must also characterize through the reduced order models $\mathcal{G}_{1r}$ and $\mathcal{G}_{2r}$ the accuracy required

of the models of each component when measured from the system of connected components. Notice that *weighted* single-component model reduction problems are defined by fixing component $\mathcal{G}_1$ (for input weighting) or component $\mathcal{G}_2$ (for output weighting). Several weighted model reduction procedures are available in the literature.

Of course, one can have more complex arrangements of components than the simple example of Figure 4(a). For instance, in Figure 4(b), a second example of multi-component model reduction involves feedback. In a typical controlled system, component $\mathcal{G}_1$ is usually the plant and component $\mathcal{G}_2$ is the controller. Both plant and controller might be subject to order reduction, where the reduced order models are represented in the picture by $\mathcal{G}_{1r}$ and $\mathcal{G}_{2r}$.

The general multi-component model reduction problem that includes all the special cases mentioned above can be loosely stated as follows

> Given N system components $\mathcal{G}_i$, $i = 1,\ldots,N$ and an interconnection architecture (as in Figures 3–4 or other), chose procedures for each component that yield reduced order models $\mathcal{G}_{ir}$, $i = 1,\ldots,N$ so that the overall system error (characterized by e in Figures 3–4) is small.

It is important to note that model reduction methods that make the single-component model reduction error small for some component in the system do not necessarily yield small errors in an interconnected architecture. Conversely, a given reduced order model of a single component might produce unbounded single-component error and small multi-component error. In other words, the architecture of components tend to have a major impact on errors and on the selection of adequate model reduction procedures. One consequence of this fact in the context of control synthesis is that the determination of the model of the plant (in this case also affected by the model reduction procedure) and the design of the control law *are not independent problems*. The recognition of this fact (see the references for examples) has led to a large research effort devoted to the integration of the disciplines of model identification and control design.

The general multi-component model reduction problem is significantly more involved than the single-component model reduction problem. Two particular cases of the multi-component model reduction problem have been extensively studied and several approaches are available in the literature: controller reduction and frequency weighted model reduction. The former is a particular case of the multi-component problem under feedback connection, depicted in Figure 4(b). The latter is a particular case of the series connection depicted in Figure 4(a).

2.3. The Quality of the Reduced Order Model

Whenever the systems $\mathcal{G}$ and $\mathcal{G}_r$ can be interpreted as operators, the norm of the difference between $\mathcal{G}$ and the reduced order model $\mathcal{G}_r$ may be a useful measure of the size of the model reduction error. In the statements of both single- and multi-component model reduction problems, the statement that the model reduction error should be kept small can be quantified through the scalar $\|\mathcal{G} - \mathcal{G}_r\|$. If such

objective is accomplished, it is expected that the error signal e resulting from the connections depicted in Figures 3 and 4 be small for input signals u in some well defined class.

Quantities other than norms can also be used in model reduction. Indeed, given a model for a physical system, it is usually possible (and sometimes very useful) to characterize it in terms of its response to certain input signals. For instance, a linear system model can be completely characterized by its impulsive response, and a number of moments (derivatives) of the impulse response evaluated at a given instant might capture important features of the original model. Hence, *matching* certain properties constitute an alternative model reduction criterion than keeping norms small. In fact, given a physical plant, it is not even necessary to have a complete model in hand to be able to perform model reduction (or identification), since the response of the system to properly generated inputs might be evaluated experimentally. Therefore, a reduced order model can be designed (identified) to match certain frequency or time response properties.

2.4. Characterization of the Single-Component Model Reduction Error

In the interest of brevity, this chapter will focus on model reduction of the plant as a single component model reduction problem. This problem is the most elementary model reduction problem and yet displays the essential mathematical concepts encountered in the more complex multi-component model reduction problem.

It will be assumed that $\mathcal{G}$ is a linear, continuous-time and time-invariant model of order $n < \infty$ described by the set of equations

$$\dot{x}(t) = Ax(t) + Bu(t),\ x(0) = 0 \tag{1}$$

$$y(t) = Cx(t) + Du(t), \tag{2}$$

where the $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input and $y(t) \in \mathbb{R}^q$ is the system output. For simplicity, the dependence of these vectors with respect to the independent variable t will be omitted whenever possible. References to the state-space realization (1–2) will frequently appear denoted by the quadruple of matrices (A, B, C, D).

The reduced order model $\mathcal{G}_r$ to be determined has the same structure as $\mathcal{G}$, that is, it is a linear, continuous-time and time-invariant model described by

$$\dot{x_r} = A_r x_r + B_r u,\ x_r(0) = 0 \tag{3}$$

$$y_r = C_r x_r + D_r u, \tag{4}$$

where the $x_r \in \mathbb{R}^{n_r}$ is the reduced order state and $y_r \in \mathbb{R}^q$ is the output of the reduced order model.

In order to emphasize the fact that these systems are linear, they will be henceforth denoted by $\mathcal{G}(s)$ and $\mathcal{G}_r(s)$, respectively. Where the complex variable s alludes to the possibility of computing a Laplace transform (frequency-domain) representation of systems (1–2) and (3–4).

The connection of $\mathcal{G}(s)$ and $\mathcal{G}_r(s)$ as in Figure 3 produces the model reduction error signal $e := y - y_r$. The relation between the common input signal u and e can be described by defining the augmented

state

$$\tilde{x} := \begin{pmatrix} x \\ x_r \end{pmatrix}, \tag{5}$$

so that the connection of the system $\mathcal{G}(s)$ and $\mathcal{G}_r(s)$ as in Figure 3 produces the linear time-invariant system

$$\dot{\tilde{x}} = \mathcal{A}\tilde{x} + \mathcal{B}u,\ \tilde{x}(0) = 0 \tag{6}$$

$$e = \mathcal{C}\tilde{x} + \mathcal{D}u \tag{7}$$

defined with the following matrices

$$\mathcal{A} := \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix}, \qquad \mathcal{B} := \begin{bmatrix} B \\ B_r \end{bmatrix}, \qquad \mathcal{C} := \begin{bmatrix} C & -C_r \end{bmatrix}, \qquad \mathcal{D} := D - D_r. \tag{8}$$

The *error system* (6–8) is denoted by $\mathcal{E}(s)$. Notice that $\mathcal{E}(s) = \mathcal{G}(s) - \mathcal{G}_r(s)$.

3. Linear System Properties

This section introduces several concepts and properties associated with linear systems that are of interest of the model reduction problem. It intends to summarize some important results that will be used in the model reduction methods to be described in Sections 4 .(See Description and Classification, System Characteristics)

3.1. Input-Output Transfer Function

Given an arbitrary input signal $u(t)$, the value of the output signal $y(t)$ of the linear system (1–2) can be calculated by the convolution integral

$$y(t) = \int_0^\infty g(t-\tau)u(\tau)d\tau \tag{9}$$

where $g(t)$ is a function that describes the response of system (1–2) to independent impulsive inputs at all input channels, that is, the linear system *impulse response*. That relation can be equivalently characterized in the somewhat simpler form

$$\mathcal{Y}(s) = \mathcal{G}(s)\mathcal{U}(s), \tag{10}$$

where $\mathcal{Y}(s)$, $\mathcal{U}(s)$ and $\mathcal{G}(s)$ denote the *Laplace transform* of, respectively, the output $y(t)$, the input $u(t)$ and the impulse response $g(t)$. In particular, it can be shown that, for the linear system (1–2), the impulse response $g(t)$ and its associated *transfer function* $\mathcal{G}(s)$ are given by

$$g(t) := Ce^{At}B + D\delta(t), \qquad \mathcal{G}(s) := C(sI - A)^{-1}B + D. \tag{11}$$

The transfer function $\mathcal{G}(s)$ is a rational function of the complex variable s and provides a frequency-domain description of the input-output behavior of the system (1–2).

From (11), more than one realization (A,B,C,D) of a linear system can produce the same impulse response $g(t)$ and transfer function $\mathcal{G}(s)$. That is, different system realizations can produce the same input-output behavior. In particular, all linear systems whose coordinates are related by

$$x = Tz, \qquad z = T^{-1}x. \tag{12}$$

where the square matrix T is nonsingular, share the same input-output transfer function. Such systems are said to be related by a *similarity transformation* which is completely characterized by matrix T. Notice that the transformed system produced by (12) has the state-space representation

$$\dot{z} = T^{-1}ATz + T^{-1}Bu,\ z(0) = 0 \tag{13}$$

$$y = CTz + Du, \tag{14}$$

and that

$$\mathcal{G}_T(s) = (CT)\left(sI - T^{-1}AT\right)^{-1}\left(T^{-1}B\right) + D = C(sI - A)^{-1}B + D = \mathcal{G}(s) \tag{15}$$

which is indeed independent of the choice of similarity transformation matrix T.

3.2. Controllability and Observability

The following concepts play an important role in the analysis of linear systems.

Definition 1 *Given the pair of matrices (A,B) where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, the following statements are equivalent:*

a) (A,B) *is* controllable,

b) There exists no scalar $\lambda \in \mathbb{C}$ and no vector $v \in \mathbb{C}^n \neq 0$ such that

$$v^*(\lambda I - A) = 0, \qquad v^*B = 0, \tag{16}$$

c) The controllability matrix

$$W_c := \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \tag{17}$$

has rank n.

Definition 2 *Given the pair of matrices (A,C) where $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{r\times n}$, the following statements are equivalent:*

a) (A,C) *is* observable,

b) There exists no scalar $\lambda \in \mathbb{C}$ and no vector $v \in \mathbb{C}^n \neq 0$ such that

$$(\lambda I - A)v = 0, \qquad Cv = 0, \tag{18}$$

c) The observability matrix

$$W_{\mathrm{o}} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{19}$$

has rank n.

By extension, a realization (A,B,C,D) is said to be controllable if the pair (A,B) is controllable and observable if the pair (A,C) is observable.

For asymptotically stable systems, controllability and observability can be equivalently checked by solving the Lyapunov equations

$$AP_{\mathrm{c}} + P_{\mathrm{c}}A^T + BB^T = 0, \tag{20}$$

$$A^T P_{\mathrm{o}} + P_{\mathrm{o}}A + C^T C = 0. \tag{21}$$

The solutions P_{c} and P_{o} are called, respectively, controllability and observability Grammians. The following lemmas are standard.

Lemma 1 *The controllability Grammian P_{c} is positive definite if, and only if, matrix A is asymptotically stable and (A,B) is controllable.*

Lemma 2 *The observability Grammian P_{o} is positive definite if, and only if, matrix A is asymptotically stable and (A,C) is observable.*

It is worth noticing that the Grammians are not realization independent since

$$P_{\mathrm{c}T} = T^{-1}P_{\mathrm{c}}T^{-T}, \qquad P_{\mathrm{o}T} := T^T P_{\mathrm{o}}T. \tag{22}$$

However, the product of the Grammians, that is,

$$P_{\mathrm{c}T}P_{\mathrm{o}T} = T^{-1}P_{\mathrm{c}}P_{\mathrm{o}}T \tag{23}$$

possess invariant eigenvalues.

3.3. Frequency Moments and Markov Parameters

Assume that the transfer function $\mathcal{G}(s)$ is strictly proper ($D = 0$) and analytic on the imaginary axis. Its Fourier power series expansion around $s = j\omega$, $0 \leq \omega < \infty$, provides

$$\mathcal{G}(s) = C(sI - A)^{-1}B = \sum_{i=0}^{\infty} M_i(j\omega)(s - j\omega)^i, \tag{24}$$

where the matrices

$$M_i(j\omega) := C(j\omega I - A)^{-(i+1)}B, \quad i = 0,1,\ldots, \tag{25}$$

are known as the *low frequency moments* of the transfer function $\mathcal{G}(s)$. The *high frequency* moments

$$M_i(j\infty) := \lim_{\omega\to\infty} M_i(j\omega) = CA^iB, \quad i = 0,1,\ldots, \tag{26}$$

can be obtained by performing a Laurent expansion around $s = 0$. The high frequency moments $M_i(j\infty)$, $i = 0,1,\ldots$ are also called *Markov parameters*. In single input systems, the Markov parameters can be given a physical interpretation by applying an unitary impulse at the input channel. Using (9–11), such input produces the output

$$y(t) = \int_0^\infty Ce^{A(t-\tau)}B\delta(\tau)d\tau = Ce^{At}B. \tag{27}$$

Therefore, Markov parameters are associated with the ith derivative (*time moment*) of the impulse response at instant zero

$$\left.\frac{d^iy(t)}{dt^i}\right|_{t=0} = M_i(j\infty), \quad i = 0,1,\ldots. \tag{28}$$

Notice that the frequency moments are input-output properties and should remain invariant under a similarity transformation. Indeed, the low frequency moments (25) are such that

$$M_i(j\omega)_T = (CT)\left(j\omega I - T^{-1}AT\right)^{-(i+1)}\left(T^{-1}B\right) = C(j\omega I - A)^{-(i+1)}B = M_i(j\omega) \tag{29}$$

for all $i = 0,1,\ldots$. The same pattern can be used to shown that the Markov parameters are also invariant.

3.4. Output Correlation and Power Moments

Another quantity related to the input-output behavior of a linear system is the deterministic *output correlation* for impulsive inputs (white noise inputs in the stochastic case). Assume that the linear model (1–2) is asymptotically stable and strictly proper ($D = 0$). The output correlation for impulsive inputs is defined by

$$R(t) = \sum_{i=1}^{m}\int_0^\infty y^i(t+\tau){y^i}^T(\tau)d\tau, \tag{30}$$

where $y^i(t)$, $i = 1,\ldots,m$ denotes the output of the system due to an impulse applied at the ith input channel. It can be shown that (30) can be computed as

$$R(t) = Ce^{At}P_cC^T, \tag{31}$$

where P_c is the controllability Grammian, i.e., the positive semidefinite solution to the Lyapunov equation (20). Following Section 3.3, the output covariance (31) can be Laplace transformed and expanded in Fourier series

$$\mathcal{R}(s) = C(sI - A)^{-1}P_cC^T = \sum_{i=0}^{\infty} R_i(j\omega)(s - j\omega)^i. \tag{32}$$

The matrices

$$R_i(j\omega) = C\,(j\omega I - A)^{-(i+1)}\,P_c C^T, i = 0,1,\ldots \tag{33}$$

are known as the *low frequency power moments*. The *high frequency* moments

$$R_i(j\infty) := \lim_{\omega\to\infty} R_i(j\omega) = CA^i P_c C^T, \quad i = 0,1,\ldots, \tag{34}$$

are called *covariance parameters*.

The same reasoning used to show that the frequency moments and Markov parameters are independent of state-space realizations can be used to show that the power moments $R_i(j\omega)$, $i = 0,1,\ldots$, are also invariant under a similarity transformation.

3.5. H_2 and H_∞ Norms

Given the transfer function $\mathcal{G}(s)$ defined in (11), the H_2 norm of $\mathcal{G}(s)$ is defined as

$$\|\mathcal{G}(s)\|_2 := \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}\left(\mathcal{G}(j\omega)\mathcal{G}^T(-j\omega)\right) d\omega\right)^{1/2}. \tag{35}$$

For a stable transfer function, Parseval's Theorem can be used to show that the above defined quantity is equivalent to

$$\|\mathcal{G}(s)\|_2 = \text{trace}\left(CP_cC^T\right)^{1/2} = \text{trace}\left(R(0)\right)^{1/2}. \tag{36}$$

That is, the H_2 norm of a transfer function $\mathcal{G}(s)$ matches the trace of its output correlation (30) evaluated at instant zero. The quantity $R(0)$ is known as the *output covariance*. Another characterization of the H_2 norm is given by a set of LMI (Linear Matrix Inequalities). For a given asymptotically stable realization $(A,B,C,0)$ it is possible to show that $\|\mathcal{G}(s)\|_2^2 < \gamma$ if, and only if, there exist symmetric matrices P and W such that the LMI

$$\begin{bmatrix} AP + PA^T & B \\ B^T & -I \end{bmatrix} < 0, \quad \begin{bmatrix} W & CP \\ PC^T & P \end{bmatrix} > 0, \quad \text{trace}\,(W) < \gamma, \tag{37}$$

hold feasible. In the above, matrix P serves as an upperbound to the controllability Grammian, that is, $P > P_c$. The matrix variables P and W appear affinely on the matrices.

Another norm of interest is defined by

$$\|\mathcal{G}(s)\|_\infty := \sup_\omega \|\mathcal{G}(j\omega)\| = \sup_\omega \bar{\sigma}(G(j\omega)). \tag{38}$$

This quantity is known as the H_∞ norm of $\mathcal{G}(s)$. For SISO systems, the H_∞ norm can be evaluated by finding the supreme value of the Bode magnitude plot[1] over all frequencies. In the case of a stable transfer function, the H_∞ norm coincides with the induced norm

$$\|\mathcal{G}(s)\|_\infty = \sup_{\int_0^\infty u^T(\tau)u(\tau)d\tau<\infty} \frac{\left(\int_0^\infty y^T(\tau)y(\tau)d\tau\right)^{1/2}}{\left(\int_0^\infty u^T(\tau)u(\tau)d\tau\right)^{1/2}} \tag{39}$$

[1] For MIMO systems, the same information can be obtained from a maximum singular value (sigma) plot.

where the signals $u(t)$ and $y(t)$ are supposed to be square integrable. For a given asymptotically stable realization (A,B,C,D) it is possible to show that $\|\mathcal{G}(s)\|_\infty^2 < \gamma$ if, and only if, there exists a symmetric matrix P such that the LMI

$$\begin{bmatrix} AP+PA^T & B & PC^T \\ B^T & -I & D^T \\ CP & D & -\gamma I \end{bmatrix} < 0, \quad P > 0, \tag{40}$$

are feasible.

3.6. The Conjugate System, Inner, Outer and All-pass Transfer Functions

Given a linear time-invariant system with transfer function (11), the system

$$\mathcal{G}^\sim(s) := \mathcal{G}^T(-s) = -B^T\left(sI+A^T\right)^{-1}C^T + D^T \tag{41}$$

is known as the *conjugate* system of $\mathcal{G}(s)$. If $\mathcal{G}^\sim(s)\mathcal{G}(s) = \alpha I$, where α is positive scalar, $\mathcal{G}(s)$ is said to be *inner*. The following lemma provides an useful characterization of an *inner* transfer function.

Lemma 3 *Given the transfer function $\mathcal{G}(s)$ defined in (11) assume that matrix A is asymptotically stable. Assume also that the observability Grammian $P_o \geq 0$ given by (21) exists. If*

$$B^T P_o + D^T C = 0 \quad \textit{and} \quad D^T D = \alpha I, \tag{42}$$

where $\alpha > 0$ then $\mathcal{G}(s)$ is inner.

Proof: The product of a linear system and its conjugate is given by

$$\mathcal{G}^\sim(s)\mathcal{G}(s) = \begin{bmatrix} D^T C & B^T \end{bmatrix} \begin{bmatrix} sI-A & 0 \\ C^T C & sI+A^T \end{bmatrix} \begin{bmatrix} B \\ -C^T D \end{bmatrix} + D^T D,$$

which is still a linear system. A similarity transformation (12) defined by the matrices

$$T = \begin{bmatrix} I & 0 \\ P_o & I \end{bmatrix}, \qquad T^{-1} = \begin{bmatrix} I & 0 \\ -P_o & I \end{bmatrix},$$

on the above system provides

$$\mathcal{G}^\sim(s)\mathcal{G}(s) = \begin{bmatrix} B^T P_o + D^T C & B^T \end{bmatrix} \begin{bmatrix} sI-A & 0 \\ A^T P_o + P_o A + C^T C & sI+A^T \end{bmatrix} \begin{bmatrix} B \\ -P_o B - C^T D \end{bmatrix} + D^T D.$$

Now, using (42) and (21)

$$\mathcal{G}^\sim(s)\mathcal{G}(s) = \begin{bmatrix} 0 & B^T \end{bmatrix} \begin{bmatrix} sI-A & 0 \\ 0 & sI+A^T \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix} + D^T D = D^T D = \alpha I$$

which shows that $\mathcal{G}(s)$ is indeed inner. □

If $\mathcal{G}(s)\mathcal{G}^{\sim}(s) = \alpha I$, then $\mathcal{G}(s)$ is said to be *co-inner*. The following lemma follows from Lemma 3 by duality.

Lemma 4 *Given the transfer function $\mathcal{G}(s)$ defined in (11) assume that matrix A is asymptotically stable. Assume also that the controllability Grammian $P_c \geq 0$ given by (20) exists. If*

$$P_c C + BD^T = 0 \quad \textit{and} \quad DD^T = \alpha I, \tag{43}$$

where $\alpha > 0$ then $\mathcal{G}(s)$ is co-inner.

A square transfer function $\mathcal{G}(s)$ which is inner (and co-inner) is said to be *all-pass*.

4. Model Reduction by Truncation

Given a linear system (1–2), a *preferred realization* will be obtained via the similarity transformation $T := \begin{bmatrix} T_1 & T_2 \end{bmatrix}$

$$\begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = x, \qquad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{bmatrix} R_1^T \\ R_2^T \end{bmatrix} x, \qquad T_1 R_1^T + T_2 R_2^T = I, \tag{44}$$

This operation transforms the coordinates of the original system into

$$\begin{pmatrix} \dot{z_1} \\ \dot{z_2} \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \tag{45}$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + Du, \tag{46}$$

where

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} := \begin{bmatrix} R_1^T A T_1 & R_1^T A T_2 \\ R_2^T A T_1 & R_2^T A T_2 \end{bmatrix}, \qquad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} := \begin{bmatrix} R_1^T B \\ R_2^T B \end{bmatrix}, \qquad \begin{bmatrix} C_1 & C_2 \end{bmatrix} := \begin{bmatrix} C T_1 & C T_2 \end{bmatrix}. \tag{47}$$

A reduced order model (3–4) is then obtained by truncating the state vector so as to preserves only the state $x_r = z_1$. That operation produces a reduced order model (3–4) with state-space realization

$$(A_r, B_r, C_r, D_r) = (A_{11}, B_1, C_1, D) = \left(R_1^T A T_1, R_1^T B, C T_1, D\right). \tag{48}$$

Notice that, by construction, $R_1^T T_1 = I$, which characterizes

$$\mathbf{P} := R_1 T_1^T \tag{49}$$

as an *oblique projection operator*. This operator is such that for any $x \in \mathcal{R}(R_1)$, $\mathbf{P}x = x$. In particular, it can be shown that all *projectors* on $\mathcal{R}(R_1)$ can be parametrized with the help of an arbitrary square and nonsingular matrix J in the form

$$T_1 = J^T R_1 \left(R_1^T J^T R_1\right)^{-1}. \tag{50}$$

One can verify that the above defined projector satisfies the properties

$$\mathbf{P}^2 = \mathbf{P}\mathbf{P} = \mathbf{P}, \qquad \mathbf{P}R_1 = R_1, \qquad \mathbf{P}^T J^T R_1 = J^T R_1. \tag{51}$$

When $\mathbf{P} = \mathbf{P}^T$ it is called an *orthogonal projector*.

In the computational component of the system referred to as the "ODE solver", suppose we want to build a simulation (or a controller) in a digital computer. The computer uses fixed point arithmetic, with beta bits in the fractional part of the wordlength, and with a uniformly distributed white noise model for round-off errors. The error in computing the state is *e*. T fixes the coordinates we choose for the calculations. G is the transfer function from input to output, and of course, it is independent of the realization (the choice of T). However, the effect of round-off error, e, is realization dependent. Thus, here is a class of problems that cannot be studied with input/output methods, such as transfer fucntions. There is no clue from the component technology, physics, how to choose the basis functions for modeling so that computational errors are small. It is clearly a waste of resources to model the physics more precisely than the computational error. Hence, an important question for all modeling problems designed for computer computations is "just how large can the round-off error be?" It has been shown by Liu and Skelton that the variance of the round-off error is unbounded over T, regardless of the wordlength. That is, one can compute in an arbitrarily bad set of coordinates. Hence, one can spend a lot of money getting good component models that are worthless is a system of interconnected components. This example shows that the manner in which one models one component of the system can affect the dynamics of another component. This illustrates the general principle that the Design and Modeling of components should not be independent decisions.

Mullis and Roberts, Williamson, and others have shown the optimal realization for digital computation, to minimize the roundoff, subject to a scaling constraint. From a component technology viewpoint, one can model the system first, and then compute the optimal realization, T. However, since T depends on the choice of the model, and the model may contain free parameters, such as material models or control parameters to be selected, the systems approach would be to jointly choose the model and the realization. Following this procedure one can design LQG controllers that are optimal for the specific finite-precision computing environment. There is no separation principle in this case, but two Riccati equations appear with a coupling term that disappears as the number of bits in the wordlength go to infinity. This theory yields controllers that are tailored to the specific computational environment, and the classical separation principle results as a special case when the computation is with infinite precision. This result has been applied in a redesign of the controller for the Hubble Space Telescope.

The original HST controller was designed assuming infinite precision computing. A redesign, using the above procedure, is accomplished , using the fact that the A/D and D/A converters have 16 bits and the control computer has 24 bits (The recent HST servicing mission changed this computer to a floating point machine). The simulation result produced 2 orders of magnitude improvement in pointing efficiency (ratio of pointing variance to control variance). This is a no cost solution, in the sense that the control complexity is not increased. Telemetry can send up new coefficients within the existing algorithm. Furthermore, using the new control design technology, one needs only 4-bit arithmatic to obtain the same performance as the existing controller achieved with 24 bits. This is perhaps not a significant contribution to either Signal Processing or Control disciplines, but the extraordinary improvement is due to integration of the disciplines.

Given a linear system in the form (1–2), the fastest and simplest way to produce a reduced order model is by truncation. For instance, a "natural" frequency domain model reduction procedure is to truncate poles and zeros. While deleting poles and zeros may be simple in SISO (Single-Input-Single-Output) systems, the many subtleties involving the definition of zeros imposes some extra difficulties for MIMO (Multiple-Input-Multiple-Output) systems. In this aspect, state-space methods seems to provide a more adequate framework, and will be the methodology of choice used throughout this chapter.

In state-space, truncation of the state vector is the "natural" choice for obtaining a reduced order model. The fundamental question is *what states are "important" and should be kept in the reduced order model*? In this section, answers to this question will be provided by transforming the original system into *preferred realizations*, where truncation of some states will be able to partially preserve selected properties of the original system. All model reduction procedures to be derived in this section follow the pattern: *transform and truncate*. Frequency domain interpretations of these results will be provided whenever available.

4.1. Minimal Transfer Equivalent Realizations

In Section 3.1, it has been shown that different realizations can have the same transfer function. This motivates the following definition.

Definition 3 *Two distinct state-space realizations of a linear time-invariant system are said to be* transfer equivalent *if they have the same transfer function, defined by (11).*

In the context of model reduction, and given a state-space realization (A,B,C,D) of order n, one might wonder whether the given realization is *minimal* in the sense that there exists no other transfer equivalent realization (A_r,B_r,C_r,D_r) of order n_r with n_r smaller than n. The answer to this question relies on the concepts of *controllability* and *observability* introduced in Section 3.2 (see also). The following lemma is standard.

Lemma 5 *The state-space realization (A,B,C,D) is minimal if, and only if, it is controllable and observable. In this case, all transfer equivalent realizations are related by a similarity transformation. The order of a minimal realization is called the* minimal degree.

The proof of this lemma can be found in the books on linear systems theory cited in the bibliography. An immediate implication is that if a given realization is not minimal, one should be able to obtain a transfer equivalent realization with reduced order. A transfer function with minimal degree is obtained when (A_r,B_r,C_r,D_r) is controllable and observable. A constructive procedure to compute such a minimal realization is based on the calculation of the controllable and observable subspaces. This is the subject addressed in the next lemmas.

Lemma 6 *Given the state-space realization (A,B,C,D) of order n and the controllability matrix W_c defined in (17), compute the singular value decomposition*

$$W_c = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 \Sigma V_1^T \tag{52}$$

where $U_1U_1^T + U_2U_2^T = I$, $V_1V_1^T + V_2V_2^T = I$ and $\Sigma = \operatorname{diag}(\sigma_1,\ldots,\sigma_{n_c}) > 0$, and calculate

$$T_1 := U_1\Sigma^{1/2}, \qquad R_1 := U_1\Sigma^{-1/2}. \tag{53}$$

The state-space realization (3–4) with (48) of order n_c is controllable.

Proof: Notice that the matrices (R_1,T_1) given in (53) define the orthogonal projector $\mathbf{P} = \mathbf{P}^T = U_1U_1^T$. Hence

$$\mathbf{P}W_c = \mathbf{P}U_1\Sigma V_1^T = U_1\Sigma V_1^T = W_c$$

or, using (17),

$$\mathbf{P}W_c = \begin{bmatrix} \mathbf{P}B & \mathbf{P}AB & \ldots & \mathbf{P}A^{n-1}B \end{bmatrix} = \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}.$$

Furthermore,

$$\begin{aligned} A\mathbf{P}B &= AB, \\ (A\mathbf{P})^2 B = A\mathbf{P}(A\mathbf{P}B) &= A\mathbf{P}AB = A^2B, \\ &\vdots \\ (A\mathbf{P})^n B = A\mathbf{P}A^{n-1}B &= A^nB, \end{aligned}$$

which implies that

$$A_r^iB_r = \left(R_1^TAT_1\right)^i \left(R_1^TB\right) = R_1^T\left(A\mathbf{P}\right)^iB = R_1^TA^iB, \quad i = 0,\ldots,n.$$

Hence, the controllability matrix of the reduced order model satisfies

$$\begin{aligned} W_{cr} &= \begin{bmatrix} B_r & A_rB_r & \ldots & A_r^{n_c-1}B_r \end{bmatrix}, \\ &= R_1^T\begin{bmatrix} B & AB & \ldots & A^{n_c-1}B \end{bmatrix}, \\ &= R_1^TW_c. \end{aligned}$$

Notice that the reduced order model is controllable since $W_{cr} = R_1^TW_c = \Sigma^{1/2}V_1^T$ has rank n_c. □

Lemma 7 *Given the state-space realization (A,B,C,D) of order n and the observability matrix W_o defined in (17), compute the singular value decomposition*

$$W_o = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 \Sigma V_1^T \tag{54}$$

where $U_1U_1^T + U_2U_2^T = I$, $V_1V_1^T + V_2V_2^T = I$ and $\Sigma = \operatorname{diag}(\sigma_1,\ldots,\sigma_{n_o}) > 0$, and calculate

$$T_1 := V_1\Sigma^{-1/2}, \qquad R_1 := V_1\Sigma^{1/2}. \tag{55}$$

The state-space realization (3–4) with (48) of order n_o is observable.

Proof: The matrices (R_1, T_1) given in (55) define the orthogonal projector $\mathbf{P} = \mathbf{P}^T = V_1 V_1^T$ so that $\mathbf{P} W_o^T = W_o^T$. Consequently, $W_{or} = W_o T_1 = U_1 \Sigma^{1/2}$, which has rank n_o. □

Lemmas 6 and 7 are able to extract, respectively, the *controllable* and *observable subspaces* of a given realization. Using these tools, a minimal transfer equivalent realization can be built in two steps. In the first step, Lemma 6 obtains the controllable subspace of (A,B,C,D). In the second step, Lemma 7 is used to build the observable subspace of the controllable system obtained in step one. The resulting system is controllable and observable. This procedure is summarized as the following algorithm.

Algorithm 1 *Given the state-space realization (A,B,C,D) of order n, follow the steps:*

Step 1. *Calculate the singular value decomposition*

$$W_c = \begin{bmatrix} U_c & U_{\bar{c}} \end{bmatrix} \begin{bmatrix} \Sigma_c & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_c^T \\ V_{\bar{c}}^T \end{bmatrix} = U_c \Sigma_c V_c^T. \tag{56}$$

where $U_c U_c^T + U_{\bar{c}} U_{\bar{c}}^T = I$, $V_c V_c^T + V_{\bar{c}} V_{\bar{c}}^T = I$ and $\Sigma_c = \mathrm{diag}\,(\sigma_{c1}, \ldots, \sigma_{cn_c}) > 0$. Define $T_c := U_c \Sigma_c^{1/2}$.

Step 2. *Calculate the singular value decomposition*

$$W_o T_c = \begin{bmatrix} U_{co} & U_{\bar{co}} \end{bmatrix} \begin{bmatrix} \Sigma_{co} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{co}^T \\ V_{\bar{co}}^T \end{bmatrix} = U_{co} \Sigma_{co} V_{co}^T \tag{57}$$

where $U_{co} U_{co}^T + U_{\bar{co}} U_{\bar{co}}^T = I$, $V_{co} V_{co}^T + V_{\bar{co}} V_{\bar{co}}^T = I$ and $\Sigma_{co} = \mathrm{diag}\,(\sigma_{co1}, \ldots, \sigma_{con_{co}}) > 0$.

Step 3. *Calculate the matrices*

$$T_1 := U_c \Sigma_c^{1/2} V_{co} \Sigma_{co}^{-1/2}, \qquad R_1 := U_c \Sigma_c^{-1/2} V_{co} \Sigma_{co}^{1/2}. \tag{58}$$

The state-space realization (3–4) with (48) of order n_{co} is a minimal transfer equivalent realization of (A,B,C,D).

Notice that in Lemmas 6, only the matrices U_1 and Σ obtained from the singular value decomposition (52) are used to build the reduced order model. Matrices V_1 and V_2 are never involved in the computations. This fact may be used to replace the singular value decomposition (52) by the symmetric eigenvalue problem

$$W_c W_c^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} = U_1 \Sigma U_1^T,$$

which may lead to improved numerical performance. The same argument can be carried over 7 to replace (54) by the symmetric eigenvalue problem of $W_o^T W_o$. These simplifications can be incorporated in Algorithm 1 with no further ado. For asymptotically stable systems, the positive semidefinite Grammians P_c and P_o can also be used instead of matrices $W_c W_c^T$ and $W_o^T W_o$ (see also Section 4.4).

Caution must be exerted when these results are to be followed by the design of a control law. It should be said that the deletion of unobservable states is always desirable (given that the output includes all variables of interest), but it is not always desirable to delete uncontrollable states. For example, one

might augment the plant dynamics of external disturbances to the plant dynamics so that deleting uncontrollable disturbances from the augmented system before control design may not be wise.

In the forthcoming sections, truncation of a realization beyond its minimal degree is pursued. Of course, there is no hope that transfer equivalence will be preserved, and some other system properties will be matched or preserved.

4.2. Matching Frequency and Power Moments

In Sections 3.3 and 3.4, it has been shown that the high and low frequency and power moments are input-output properties that remain invariant under a similarity transformation. Therefore, the minimal transfer equivalent realization obtained in Section 4.1, which preserves the transfer functions, also preserves these parameters. In this section, reduced order models that do not preserve transfer function equivalence will be obtained by matching a subset of the frequency and power moments.

One can ask the following question. "How much information about the plant do we really need to compute the optimal control?" If we use the separation principle in the finite time LQG problem, we need the entire state space model. Yet, in the problem shown in Figure 2, only the first N Markov parameters are needed to compute the exact optimal control. Modeling from first principles, physics, focuses on modeling A and B and C, over-parametrizing the model by a large margin. In a 1000th order system, there are more than a million parameters in A, B, and C. It is only a special combination of these parameters that is important to controlled performance. The use of a separation principle seems to utilize much more information about a model than is really necessary. Note that any model obtained from data that matches the first N Markov parameters will produce the exact optimal control for the finite horizon optimal control problem, regardless what other properties the model possesses. There are several identification methods which can preserve the first N Markov parameters. Since Markov parameters can be obtained from data, one can eliminate the step that computes the markov parameters and compute the optimal control directly from data.

To illustrate how many models might be equivalent to a given set of data. Suppose we compute the first q auto-correlations and the first q cross-correlations, from the real data, (which of course evolves from a nonlinear system). Then we can construct the matrix R and H as following.

$$R_i = \mathcal{E}\left[y_{k+i}y_k^T\right] \ , \ H_i = \mathcal{E}\left[y_{k+i}u_k^T\right]$$

where $\mathcal{E}[\cdot]$ is the expectation operator.

$$R = \begin{bmatrix} R_0 & R_1^T & R_2^T & R_3^T \\ R_1 & R_0 & R_1^T & R_2^T \\ R_2 & R_1 & R_0^T & R_1^T \\ R_3 & R_2 & R_1 & R_0 \end{bmatrix} \ , \ H = \begin{bmatrix} H_0 & H_1^T & H_2^T & H_3^T \\ H_1 & H_0 & H_1^T & H_2^T \\ H_2 & H_1 & H_0^T & H_1^T \\ H_3 & H_2 & H_1 & H_0 \end{bmatrix}$$

"Does there exist any linear model that can match these $2q$ pieces of data exactly?" The answer is none if the matrix $R - HH$ has a negative eigenvalue, and an infinite number otherwise. (Note that only one of these models is the exact model derived from physics). There is a q Markov COVER

algorithm (matches the first q covariance parameters and the first q Markov parameters) to generate all models that can match this data. Note that any of these infinite number of models (q Markov COVERS) will yield the exact optimal control for an LQG problem with horizon equal to q.

The basic tool is the use of projectors. The next lemma shows how to develop reduced order models that match a subset of the high frequency and power moments, that is, the Markov and the covariance parameters.

Lemma 8 *Given the minimal and asymptotically stable realization $(A,B,C,0)$ of order n, compute the symmetric and positive definite controllability Grammian P_c. Calculate the singular value decomposition*

$$W_q = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 \Sigma V_1^T, \tag{59}$$

where $U_1U_1^T + U_2U_2^T = I$, $V_1V_1^T + V_2V_2^T = I$ and $\Sigma = \mathrm{diag}\,(\sigma_1,\ldots,\sigma_{n_r}) > 0$, where

$$W_q := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix}. \tag{60}$$

Calculate

$$T_1 := P_c V_1 \left(V_1^T P_c V_1\right)^{-1}, \qquad R_1 := V_1, \tag{61}$$

The state-space realization (3–4) with (48) of order n_r is asymptotically stable and matches the first q Markov $M_i(j\infty)$ and the first q covariance parameters $R_i(j0)$, $i = 0,\ldots,q-1$, of $(A,B,C,0)$.

Proof: The matrices R_1 and T_1 define the projector $\mathbf{P} = R_1T_1^T$ such that[2]

$$\mathbf{P}W_q^T = W_q^T, \qquad \mathbf{P}^T P_c W_q^T = P_c W_q^T.$$

Proceeding as in the proof of Lemma 6 one can show that

$$W_q\mathbf{P}^T = W_q \quad \Rightarrow \quad C_rA_r^i = CA^iT_1, \quad i = 0,\ldots,q$$

so that

$$W_{q_r} = \begin{bmatrix} C_r \\ C_rA_r \\ \vdots \\ C_rA_r^{q-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix} T_1 = W_qT_1.$$

Furthermore, one can verify that the reduced order model satisfies

$$A_rP_{cr} + P_{cr}A_r^T + B_rB_r^T = 0,$$

[2]The free parameter J in (50) is set to P_c.

where the controllability Grammian $P_{cr} = R_1^T P_c R_1$. Consequently

$$\begin{aligned} W_{q_r}\begin{bmatrix} B_r & P_{cr}C_r^T \end{bmatrix} &= \begin{bmatrix} W_q T_1 R_1^T B & W_q T_1 R_1^T P_c R_1 T_1^T C^T \end{bmatrix}, \\ &= \begin{bmatrix} W_q \mathbf{P}^T B & W_q \mathbf{P}^T P_c \mathbf{P} C^T \end{bmatrix}, \\ &= \begin{bmatrix} W_q B & W_q P_c \mathbf{P} C^T \end{bmatrix}, \\ &= W_q \begin{bmatrix} B & P_c C^T \end{bmatrix}. \end{aligned}$$

Notice that

$$W_q \begin{bmatrix} B & P_c C^T \end{bmatrix} = \begin{bmatrix} M_0(j\infty) & R_0(j\infty) \\ M_1(j\infty) & R_1(j\infty) \\ \vdots & \vdots \\ M_q(j\infty) & R_q(j\infty) \end{bmatrix},$$

which guarantees the matching of the desired parameters. The stability of the reduced order system comes from the minimality of the original system. Indeed, the observability Grammian of the reduced order system is $P_{or} := T_1^T P_o T_1$, where P_o is the observability Grammian of the original system. Notice that (A,B) being controllable implies $P_c > 0$ which implies that T_1 is nonsingular. Now (A,C) being observable implies that $P_o > 0 \Rightarrow P_{or} > 0$. Since $W_{or} = W_{q_r} \Rightarrow \text{rank}\,(W_{or}) = n_r$ the reduced order system is certainly observable and Lemma 2 ensures that A_r is asymptotically stable. □

According to Section 3.3, matching the first q Markov parameters guarantee that the first q time moments of the impulse response are matched. This guarantees that the time-response of the reduced order model will have the "right start". The preservation of this feature is specially important in *nonminimum phase* systems. For instance, the response of a nonminimum phase system to a positive step might present at time $t = 0$ a negative derivative. This behavior can be captured by matching high frequency moments (see Section 5.1 for an example). Low frequency parameters can be matched through the following lemma.

Lemma 9 *Given the minimal and asymptotically stable realization $(A,B,C,0)$ of order n, compute the symmetric and positive definite controllability Grammian P_c. Calculate the singular value decomposition*

$$W_p = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 \Sigma V_1^T, \tag{62}$$

where $U_1U_1^T + U_2U_2^T = I$, $V_1V_1^T + V_2V_2^T = I$ and $\Sigma = \text{diag}\,(\sigma_1, \ldots, \sigma_{n_r}) > 0$, where

$$W_p := \begin{bmatrix} CA^{-1} \\ CA^{-2} \\ \vdots \\ CA^{-p} \end{bmatrix}. \tag{63}$$

Calculate

$$T_1 := P_c V_1 \left(V_1^T P_c V_1\right)^{-1}, \qquad R_1 := V_1, \tag{64}$$

If the matrix A_r obtained in the state-space realization (3–4) with (48) of order n_r is nonsingular, then the reduced order system is asymptotically stable and matches the first p low frequency moments $M_i(j0)$ and the first p power moments $R_i(j0)$, $i = 0, \ldots, p-1$ of $(A,B,C,0)$.

Proof: The proof of this lemma follows the same pattern as the one of Lemma 8 since $\mathbf{P}W_p^T = W_p^T$ and $\mathbf{P}^T P_c W_p^T = P_c W_p^T$. When $A_r = R_1^T A T_1$ is nonsingular

$$CA^{-1}T_1 = CA^{-1}T_1A_rA_r^{-1} = CA^{-1}T_1R_1^TAT_1A_r^{-1} = CA^{-1}\mathbf{P}^TAT_1A_r^{-1} = CA^{-1}AT_1A_r^{-1} = C_rA_r^{-1}.$$

Working recursively as in the proof of Lemma 6, one can show that

$$W_p\mathbf{P}^T = W_p \quad \Rightarrow \quad C_rA_r^{-i} = CA^{-i}T_1, \quad i = 0,\ldots,p$$

The rest of the proof is identical to the proof of Lemma 8. Notice that when A_r is nonsingular the observability matrix $W_{or} = W_{p_r}A_r^p$ has rank n_r, which ensures the stability of the reduced order system. □

When matrix A_r is singular the obtained reduced order model is not guaranteed to match any low frequency moments. Although in most cases A_r will be nonsingular, it does not seem to be easy to show that the proposed projection will always provide a nonsingular matrix A_r. Also notice that, in general, $A_r^{-1} \neq R_1^TA^{-1}T_1$.

Matching low frequency moments guarantee that the steady-state values will be preserved. For instance, the steady-state value to a step response is matched when the first low frequency moment is matched (see Section 5.1 for an example). High and low frequency moments can be simultaneously matched by combining Lemmas 8 and 9 as in the next algorithm.

Algorithm 2 *Given the minimal and asymptotically stable state-space realization $(A,B,C,0)$ of order n, follow the steps:*

Step 1. *Calculate the singular value decomposition*

$$\begin{bmatrix} W_q \\ W_p \end{bmatrix} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1\Sigma V_1^T, \tag{65}$$

where $U_1U_1^T + U_2U_2^T = I$, $V_1V_1^T + V_2V_2^T = I$ and $\Sigma = \mathrm{diag}\,(\sigma_1,\ldots,\sigma_{n_r}) > 0$, and matrices W_q and W_p are defined, respectively by (59) and (63).

Step 2. *The matrices*

$$T_1 := P_cV_1\left(V_1^TP_cV_1\right)^{-1}, \qquad R_1 := V_1, \tag{66}$$

produce the state-space realization (3–4) with (48) of order n_r. If the reduced order system has a nonsingular matrix A_r, then it is asymptotically stable and matches the first p low frequency moments $M_i(j0)$ and power moments $R_i(j0)$, $i = 0,\ldots,p-1$ and the first q high frequency moments $M_i(j\infty)$ and power moments $R_i(j\infty)$, $i = 0,\ldots,q-1$ of $(A,B,C,0)$.

These results can be extended to cope with moments evaluated at any set of finite frequencies. It is interesting to notice that the projections generate reduced order models that are not guaranteed to approximate the original system according to any system norm. On the other hand, it gives the engineer the opportunity to *tune* up the reduced order model to an specific application by arbitrarily selecting the appropriate sets of frequencies and the order of moments that are significant to the problem. Furthermore, as the examples in Section 5 illustrate, this does not prevent the model reduction error to have a small norm.

4.3. Component Cost Analysis

Several system and control problems are formulated with the objective of monitoring or controlling the system output covariance (see Sections 3.4 and 3.5). The trace of the output covariance is referred in this section simply as the *cost function*. When the system disturbance inputs are modeled as impulses (white noise), these problems can be seen as equivalent to the minimization of the H_2 norm of the transfer function from the disturbances to the outputs, as discussed in Section 3.4. This justifies the need to develop reduced order models that are able to preserve the H_2 norm of the original system. The analysis of this model reduction problem is significantly simplified when the original system is transformed into the particular set of coordinates defined below.

Definition 4 *The asymptotically stable and time-invariant state-space realization* $(A,B,C,0)$ *of order* n*, is said to be in* cost decoupled coordinates *if the* H_2 *norm of the transfer function* $\mathcal{G}(s)$ *(cost function) can be expressed in the form* $\|\mathcal{G}(s)\|_2 = \left(\sum_{i=1}^{n} \alpha_i\right)^{1/2}$ *where the scalars* $\alpha_i > 0$*,* $i = 1,\ldots,n$ *represent* independent *contributions of each state to the cost function.*

For a given linear system, there exists more than one realization whose coordinates qualify as cost decoupled coordinates. The following lemma provide two of these realizations.

Lemma 10 *An asymptotically stable and time-invariant realization* $(A,B,C,0)$ *where*

a) the controllability Grammian P_c *given in (20) is diagonal, or*

b) the matrix $P_c C^T C$ *is diagonal*

is in cost decoupled coordinates.

Proof: The proof of this lemma consists in finding the independent factors α_i mentioned in Definition 4. In a), matrix P_c is diagonal, therefore

$$\|\mathcal{G}(s)\|_2^2 = \text{trace}\left(CP_cC^T\right) = \sum_{i=1}^{n} (P_c)_{ii}\, c_i c_i^T = \sum_{i=1}^{n} \alpha_i$$

where $\alpha_i := (P_c)_{ii} c_i c_i^T$ and the row vector c_i denotes the ith row of matrix C. In b), matrix $P_c C^T C$ is diagonal and the cost function can be calculated as

$$\|\mathcal{G}(s)\|_2^2 = \text{trace}\left(P_cC^TC\right) = \sum_{i=1}^{n} \left(P_cC^TC\right)_{ii} = \sum_{i=1}^{n} \alpha_i$$

where $\alpha_i := \left(P_cC^TC\right)_{ii}$. □

The next lemma shows how to transform a given realization into a realization with cost decoupled coordinates that simultaneously satisfies items $a)$ and $b)$ of Lemma 10. A realization with cost decoupled coordinates that only satisfies item $a)$ will be introduced in Section 4.4.

Lemma 11 *Given the controllable and asymptotically stable realization $(A,B,C,0)$ of order n, compute the symmetric and positive definite controllability Grammian P_{c}. Calculate a nonsingular matrix $F \in \mathbb{R}^{n\times n}$ such that*

$$P_{\mathrm{c}} = F^T F, \tag{67}$$

and the singular value decomposition

$$FC^T CF^T = U\Lambda U^T, \tag{68}$$

where $UU^T = I$ and $\Lambda = \operatorname{diag}(\alpha_1,\ldots,\alpha_n) \geq 0$. The similarity transformation (12) where

$$T := F^T U \tag{69}$$

puts the original realization in cost decoupled coordinates.

Proof: Since the original realization is assumed to be controllable and asymptotically stable, the controllability Grammian P_{c} is positive definite, which ensures the existence of the nonsingular factor F which can be obtained, for instance, by a Cholesky decomposition. Furthermore

$$\begin{aligned} P_{\mathrm{c}T} = T^{-1}P_{\mathrm{c}}T^{-T} &= \left(U^T F^{-T}\right)\left(F^T F\right)\left(F^{-1}U\right) = I, \\ P_{\mathrm{c}T}C_T^T C_T = T^T C^T CT &= U^T FC^T CF^T U = \Lambda, \end{aligned}$$

which shows that the transformed system satisfies both items $a)$ and $b)$ of Lemma 10. □

For model reduction, the previous lemma can be used to transform the original system into cost-decoupled coordinates while truncation follows as in the following algorithm.

Algorithm 3 *Given the asymptotically stable and controllable time-invariant state-space realization $(A,B,C,0)$ of order n follow the steps:*

Step 1. *Calculate the controllability Grammian P_{c} and compute matrix $F \in \mathbb{R}^{n\times n}$ as in (67).*

Step 2. *Compute the singular value decomposition*

$$F^T CC^T F = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix}, \tag{70}$$

where $U_1U_1^T + U_2U_2^T = I$, $\Lambda_1 = \operatorname{diag}(\alpha_1,\ldots,\alpha_{n_r}) \geq 0$ and $\Lambda_2 = \operatorname{diag}(\alpha_{n_r+1},\ldots,\alpha_n) \geq 0$.

Step 3. *The matrices*

$$T_1 := F^T U_1, \qquad\qquad R_1 := F^{-1}U_1, \tag{71}$$

produce the reduced order state-space realization (3–4) with (48) of order n_r.

Because the truncation is performed in cost decoupled coordinates, where the cost function is constituted of *independent contributions* from each state, the reduced order model obtained by the above algorithm has the following properties.

Theorem 12 *Consider the reduced order model of order n_r produced by Algorithm 3. The following statements are true.*

a) The reduced order model is asymptotically stable if and only if (A_r, B_r) is controllable.

b) If the reduced order model is asymptotically stable, the H_2 norm of the reduced order model is such that

$$\|\mathcal{G}_r(s)\|_2 = \left(\sum_{i=1}^{n_r} \alpha_i\right)^{1/2}, \tag{72}$$

c) If the reduced order model is asymptotically stable, the difference between the H_2 norm of the original system and H_2 norm of the reduced order model is such that

$$\|\mathcal{G}(s)\|_2^2 - \|\mathcal{G}_r(s)\|_2^2 = \sum_{i=n_r+1}^{n} \alpha_i, \tag{73}$$

Proof: Assume that the realization $(A,B,C,0)$ has already been transformed into cost decoupled coordinates through Lemma 11. In these coordinates, $P_{\mathrm{c}} = I$ such that

$$A + A^T + BB^T = 0 \quad \Rightarrow \quad A_{11} + A_{11}^T + B_1 B_1^T = 0.$$

That is, $P_{\mathrm{c}r} = I$. Hence, according to Lemma 1, A_{11} is asymptotically stable if, and only if, $(A_r, B_r) = (A_{11}, B_1)$ is controllable, which proves item a). Item b) comes from

$$\|\mathcal{G}_r(s)\|_2^2 = \mathrm{trace}\left(C_r P_{\mathrm{c}r} C_r^T\right) = \mathrm{trace}\left(C_r^T C_r\right) = \mathrm{trace}\left(C_1^T C_1\right) = \sum_{i=0}^{n_r} \alpha_i.$$

Notice that b) immediately implies that

$$\|\mathcal{G}(s)\|_2^2 - \|\mathcal{G}_r(s)\|_2^2 = \sum_{i=1}^{n} \alpha_i - \sum_{i=0}^{n_r} \alpha_i = \sum_{i=n_r+1}^{n} \alpha_i,$$

which is item c). □

Unfortunately, the above lemma does not guarantee any upperbound on the norm of the model reduction error $\mathcal{E}(s) = \mathcal{G}(s) - \mathcal{G}_r(s)$. This fact will be overcome in the Section 4.4. A lowerbound on the H_2 norm of the model reduction error is available

$$\|\mathcal{G}(s) - \mathcal{G}_r(s)\|_2 \geq \|\mathcal{G}(s)\|_2 - \|\mathcal{G}_r(s)\|_2 = \left(\sum_{i=1}^{n} \alpha_i\right)^{1/2} - \left(\sum_{i=1}^{n_r} \alpha_i\right)^{1/2}. \tag{74}$$

The expression above comes by combining item b) of Lemma 12 and a generic property of norms.

It is important to stress that the use of the results of this section require the careful intervention of the engineer in the selection of the system output. In the most common situations, an engineering system will have less outputs than its order. Hence, the square matrix $P_c C^T C$ will be, typically, rank deficient. For single-output systems, for example, it will have rank one and, regardless of the order of the original system, Theorem 12 tells that Algorithm 3 is able to produce a reduced order model of order one that perfectly matches the H_2 norm of the original system ($\alpha_i = 0$, $i = 2, \ldots, n$). This fact reinforces the need for a good selection of the system output, which should be able to capture the important aspects of the system behavior. An appropriate output for model reduction will often be constituted by the measured output of the original system augmented by some of its derivatives[3] and other selected important signals which are internal to the model.

It is interesting to notice that it is also possible to generate a reduced order systems that matches exactly the H_2 norm of the original system by extracting from this system its inner component. Indeed, a standard result in linear systems theory lets one writes any minimal and asymptotically stable transfer function $\mathcal{G}(s)$ in the factored form

$$\mathcal{G}(s) = \mathcal{G}_i(s)\mathcal{G}_o(s),$$

where $\mathcal{G}_i(s)$ is asymptotically stable and inner ($\mathcal{G}_i(s)^{\sim}\mathcal{G}_i(s) = I$) and $\mathcal{G}_o(s)$ is asymptotically stable and minimum phase (outer). Notice that

$$\begin{aligned} \|\mathcal{G}(s)\|_2^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}\left(\mathcal{G}_o^T(-j\omega)\mathcal{G}_i^T(-j\omega)\mathcal{G}_i(j\omega)\mathcal{G}_o(j\omega)\right) d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}\left(\mathcal{G}_o^T(-j\omega)\mathcal{G}_o(j\omega)\right) d\omega = \|\mathcal{G}_o(s)\|_2^2 \end{aligned}$$

and that $\mathcal{G}_o(s)$ is usually of reduced order.

4.4. Balanced Realization and Truncation

This section discuss one of the the most celebrated truncation methods for model reduction. It starts investigating the possibility of transforming a stable realization (A,B,C,D) into a privileged set of coordinates, known as *balanced coordinates*.

Definition 5 *The asymptotically stable and time-invariant state-space realization (A,B,C,D) of order n, is said to be in* balanced coordinates *if the controllability and observability Grammians are both diagonal.*

Under the similarity transformation (12), the controllability and observability Grammians are transformed as

$$P_{cT} := T^{-1}P_c T^{-T}, \qquad P_{oT} := T^T P_o T, \qquad P_{cT}P_{oT} := T^{-1}P_c P_o T^T. \tag{75}$$

[3] See Section 4.2 for a method that matches frequency moments and Section 3.3 for the relation between frequency moments and time moments.

Hence, it is possible to transform a given realization into balanced coordinates whenever there exists a nonsingular matrix T that simultaneously diagonalize (75). For minimal realizations, the next lemma shows in a constructive way that this is always possible.

Lemma 13 *Given the minimal and asymptotically stable realization (A,B,C,D) of order n, compute the symmetric and positive definite controllability and observability Grammians P_c and P_o. Calculate a nonsingular matrix $F \in \mathbb{R}^{n\times n}$ such that*

$$P_c = F^T F, \tag{76}$$

and the singular value decomposition

$$F P_o F^T = U \Sigma U^T, \tag{77}$$

where $UU^T = I$ and $\Sigma = \operatorname{diag}(\sigma_1,\ldots,\sigma_n) > 0$. The similarity transformation (12) where

$$T := F^T U \Sigma^{-1/4} \tag{78}$$

puts the original realization in balanced coordinates.

Proof: Since the original realization is assumed to be minimal and asymptotically stable, the Grammians P_c and P_o are positive definite, which ensures the existence of nonsingular matrices F and Σ. Furthermore one can verify that

$$T^{-1}P_c T^{-T} = \Sigma^{1/2}, \qquad T^T P_o T = \Sigma^{1/2}, \qquad T^{-1}P_c P_o T = \Sigma. \tag{79}$$

which shows that the transformed system is in balanced coordinates. □

Notice that since P_c is diagonal, balanced coordinates are also cost decoupled coordinates. Truncation of the original model is performed as in the next algorithm.

Algorithm 4 *Given the minimal and asymptotically stable realization (A,B,C,D) of order n, follow the steps:*

Step 1. *Calculate the controllability Grammian P_c and compute matrix $F \in \mathbb{R}^{n\times n}$ as in (76).*

Step 2. *Calculate the observability Grammian P_o and the singular value decomposition*

$$F P_o F^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix}, \tag{80}$$

where $U_1U_1^T + U_2U_2^T = I$, $\Sigma_1 = \operatorname{diag}(\sigma_1 I,\ldots,\sigma_r I) > 0$, $\Sigma_2 = \operatorname{diag}(\sigma_{r+1}I,\ldots,\sigma_p I) > 0$ and $\sigma_i \neq \sigma_j$ for all $i \neq j$. The singular value σ_i is assumed to have multiplicity m_i so that $\sum_{i=1}^{p} m_i = n$ and $\sum_{i=1}^{r} m_i = n_r$. Partition matrix C in p blocks of columns according to the blocks of Σ_1 and Σ_2 and calculate the component costs

$$\alpha_i = \sigma_i^{1/2} \operatorname{trace}\left(c_i c_i^T\right), \quad i = 1,\ldots,p. \tag{81}$$

Step 3. *The matrices*

$$T_1 := F^T U_1 \Sigma_1^{-1/4}, \qquad R_1 := F^{-1} U_1 \Sigma_1^{1/4}, \tag{82}$$

produce the reduced order state-space realization (3–4) with (48) of order n_r.

Notice the similarities between the above algorithm and Algorithm 3. At least for discrete-time systems, the latter can be generated as a particular case of the former by defining the infinite dimensional output vector

$$\tilde{C} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix}.$$

In this case, (80) and (70) become identical since $P_o = \sum_{i=0}^{\infty} (CA^i)^T (CA^i) = \tilde{C}^T \tilde{C}$.

The next theorem summarizes the properties of the obtained reduced order model.

Theorem 14 *Consider the reduced order model of order* n_r *produced by Algorithm 4. The following statements are true.*

a) The reduced order model is asymptotically stable.

b) The H_∞ *norm of the model reduction error is such that*

$$\| \mathcal{G}(s) - \mathcal{G}_r(s) \|_\infty \leq 2 \sum_{i=r+1}^{p} \sigma_i^{1/2}. \tag{83}$$

c) The H_2 *norm of the model reduction error is such that*

$$\| \mathcal{G}(s) - \mathcal{G}_r(s) \|_2 \leq 2 \sum_{i=r+1}^{p} \left(\sigma_i \sum_{j=1}^{i} \frac{\alpha_j}{\sigma_j} - \alpha_i \right)^{1/2}. \tag{84}$$

Proof: For simplicity, it is assumed throughout this proof that the original system has already been transformed into balanced coordinates.

a) For the moment, consider the *one step* model reduction, that is, assume that Algorithm 4 has produced a reduced order model $\mathcal{G}_r^p(s)$ by truncating only the singular value σ_p. In this case, $r = (p-1)$ and $\Sigma_2 = \sigma_p I$. Furthermore, the original model satisfies the Lyapunov equations

$$A_{11}\Sigma_1^{1/2} + \Sigma_1^{1/2} A_{11}^T + B_1 B_1^T = 0, \qquad A_{11}^T \Sigma_1^{1/2} + \Sigma_1^{1/2} A_{11} + C_1^T C_1 = 0, \tag{85}$$

$$A_{21}\Sigma_1^{1/2} + \sigma_p^{1/2} A_{12}^T + B_2 B_1^T = 0, \qquad A_{12}^T \Sigma_1^{1/2} + \sigma_p^{1/2} A_{21} + C_2^T C_1 = 0, \tag{86}$$

$$\sigma_p^{1/2} \left(A_{22} + A_{22}^T \right) + B_2 B_2^T = 0, \qquad \sigma_p^{1/2} \left(A_{22} + A_{22}^T \right) + C_2^T C_2 = 0. \tag{87}$$

Let $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$ be such that $A_{11}v = \lambda v$. Multiplying the second equation in (85) by v on the right and by v^* on the left one obtains

$$(\lambda + \lambda^*)\, v^* \Sigma_1^{1/2} v = v^* \Sigma_1^{1/2} A_{11} v + v^* A_{11}^T \Sigma_1^{1/2} v = -v^* C_1^T C_1^T v \leq 0.$$

Since $\Sigma_1 > 0$, this implies that $\mathrm{Re}\,(\lambda) \leq 0$, that is, matrix A_{11} should be asymptotically stable or, in the worst case, it has one or more eigenvalues on the imaginary axis. Hence, in order to prove the asymptotic stability of the reduced order model, it suffices to show that A_{11} has no eigenvalues on the imaginary axis. For that sake, assume that A_{11} has some eigenvalue $j\omega$ on the imaginary axis and denote by $\mathcal{V}$ the set of eigenvectors associated with this eigenvalue. Since Σ_1 and σ_p are nonsingular, the equations on the right of (85–86) can be rewritten as

$$A_{11}\left(\sigma_p \Sigma_1^{-1/2}\right) + \left(\sigma_p \Sigma_1^{-1/2}\right) A_{11}^T + \sigma_p \Sigma_1^{-1/2} C_1^T C_1 \Sigma_1^{-1/2} = 0, \tag{88}$$

$$A_{21}\left(\sigma_p \Sigma_1^{-1/2}\right) + \sigma_p^{1/2} A_{12}^T + \sigma_p^{1/2} C_2^T C_1 \Sigma_1^{-1/2} = 0. \tag{89}$$

Equations (85–87) and (88–89) can then be added

$$A_{11}\Sigma_+ + \Sigma_+ A_{11}^T + B_1 B_1^T + \sigma_p \Sigma_1^{-1/2} C_1^T C_1 \Sigma_1^{-1/2} = 0, \tag{90}$$

$$A_{21}\Sigma_+ + 2\sigma_p^{1/2} A_{12}^T + B_2 B_1^T + \sigma_p^{1/2} C_2^T C_1 \Sigma_1^{-1/2} = 0, \tag{91}$$

$$2\sigma_p^{1/2}\left(A_{22} + A_{22}^T\right) + B_2 B_2^T + C_2^T C_2 = 0, \tag{92}$$

or subtracted

$$A_{11}\Sigma_- + \Sigma_- A_{11}^T + B_1 B_1^T - \sigma_p \Sigma_1^{-1/2} C_1^T C_1 \Sigma_1^{-1/2} = 0, \tag{93}$$

$$A_{21}\Sigma_- + B_2 B_1^T - \sigma_p^{1/2} C_2^T C_1 \Sigma_1^{-1/2} = 0, \tag{94}$$

$$B_2 B_2^T - C_2^T C_2 = 0, \tag{95}$$

where

$$\Sigma_+ := \left(\Sigma_1^{1/2} + \sigma_p \Sigma_1^{-1/2}\right), \qquad \Sigma_- := \left(\Sigma_1^{1/2} - \sigma_p \Sigma_1^{-1/2}\right).$$

Notice that $\Sigma_+ > 0$ and that Σ_- is nonsingular since no diagonal entry in Σ_1 equals σ_p. Now, select a nonzero vector $z \in \mathcal{V}$, that is,

$$(A_{11} - j\omega I)\, z = 0 \quad \Rightarrow \quad z^*\left(A_{11}^T + j\omega I\right) = 0.$$

Equation (93) can be rewritten in the form

$$(A_{11} - j\omega I)\,\Sigma_- + \Sigma_-\left(A_{11}^T + j\omega I\right) + B_1 B_1^T - \sigma_p \Sigma_1^{-1/2} C_1^T C_1 \Sigma_1^{-1/2} = 0, \tag{96}$$

which right multiplied by $v := \Sigma_-^{-1} z$ on the right and by v^* on the left provides

$$v^*\left(B_1 B_1^T - \sigma_p \Sigma_1^{-1/2} C_1^T C_1 \Sigma_1^{-1/2}\right) v = 0 \quad \Rightarrow \quad B_1^T v = 0, \quad C_1 \Sigma_1^{-1/2} v = 0. \tag{97}$$

Now, the multiplication of (94) on the right by v using (97) provides

$$A_{21}\Sigma_- v = A_{21} z = 0.$$

Hence, there exists

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} z \\ 0 \end{pmatrix} = j\omega \begin{pmatrix} z \\ 0 \end{pmatrix}.$$

which implies that A also has $j\omega$ as an eigenvalue. This contradicts the hypothesis that A is asymptotically stable and shows that A_{11} must have no eigenvalues on the imaginary axis, that is, A_{11} is asymptotically stable. By noticing that the reduced order model $\mathcal{G}_r^p(s)$ is asymptotically stable and that it is still in balanced coordinates, a one step model reduction can again be used to obtain $\mathcal{G}_r^{(p-1)}(s)$ which, following the same reasoning as above, should also be asymptotically stable. The desired asymptotically stable reduced order model $\mathcal{G}_r(s)$ is then obtained by applying as many successive one step model reductions as necessary.

b) Consider the reduced order model $\mathcal{G}_r^p(s)$ obtained by a one step application of Algorithm 4. The model reduction error $\mathcal{E}_p(s) := \mathcal{G}(s) - \mathcal{G}_r^p(s)$ is given by (6–8) with $\mathcal{D} = D - D_r = 0$. The equations (90–95) can be used to show that

$$\hat{A}\hat{P}_c + \hat{P}_c\hat{A}^T + \hat{B}\hat{B}^T = 0, \tag{98}$$

where

$$\hat{A} := \mathcal{A}, \qquad \hat{B} := \begin{bmatrix} \mathcal{B} & \bar{B} \end{bmatrix},$$

$$\hat{P}_c := \begin{bmatrix} \Sigma_+ & 0 & \Sigma_- \\ 0 & 2\sigma_p^{1/2} I & 0 \\ \Sigma_- & 0 & \Sigma_+ \end{bmatrix}, \qquad \bar{B} := \begin{bmatrix} \sigma_p^{1/2}\Sigma_1^{-1/2} C_1^T \\ C_2^T \\ -\sigma_p^{1/2}\Sigma_1^{-1/2} C_1^T \end{bmatrix}.$$

Additionally, one can define matrices

$$\hat{C} := \begin{bmatrix} \mathcal{C} \\ -\bar{C} \end{bmatrix}, \qquad \bar{C} := \begin{bmatrix} \sigma_p^{1/2} B_1^T \Sigma_1^{-1/2} & B_2^T & \sigma_p^{1/2} B_1^T \Sigma_1^{-1/2} \end{bmatrix}, \qquad \hat{D} := \begin{bmatrix} 0 & 2\sigma_p^{1/2} \\ 2\sigma_p^{1/2} & 0 \end{bmatrix},$$

such that

$$\hat{P}_c\hat{C}^T + \hat{B}\hat{D}^T = 0.$$

From Lemma 4, the system

$$\hat{\mathcal{E}}_p(s) := \hat{C}\left(sI - \hat{A}\right)^{-1}\hat{B} + \hat{D}$$

is all-pass, that is, $\hat{\mathcal{E}}_p(s)\hat{\mathcal{E}}_p^{\sim}(s) = \hat{D}\hat{D}^T = 4\sigma_p I$. Notice that the relation

$$\mathcal{E}_p(s) = \begin{bmatrix} I & 0 \end{bmatrix} \hat{\mathcal{E}}_p(s) \begin{bmatrix} I \\ 0 \end{bmatrix}$$

characterizes the system $\hat{\mathcal{E}}_p(s)$ as a *dilation* of the error system $\mathcal{E}_p(s)$. Therefore, it is possible to establish the following inequality

$$\|\mathcal{E}_p(s)\|_\infty \leq \|\hat{\mathcal{E}}_p(s)\|_\infty = 2\sigma_p^{1/2}.$$

As in the proof of item *a*), successive one step model reduction operations can be applied so that, at each iteration, the bound on the error

$$\|\mathcal{E}_i(s)\|_\infty = \|\mathcal{G}_r^{i+1}(s) - \mathcal{G}_r^i(s)\|_\infty \leq \|\hat{\mathcal{E}}_i(s)\|_\infty = 2\sigma_i^{1/2}$$

is available. The bound on the fully reduced order model can be obtained by noticing that

$$\|\mathcal{G}(s)-\mathcal{G}_r(s)\|_\infty = \|\sum_{i=r+1}^{p}\mathcal{E}_i(s)\|_\infty \leq \sum_{i=r+1}^{p}\|\mathcal{E}_i(s)\|_\infty \leq 2\sum_{i=r+1}^{p}\sigma_i^{1/2}.$$

c) Consider once more the one step model reduction error $\mathcal{E}_p(s)$ and evaluate the quantity

$$\begin{aligned}\text{trace}\left(\mathcal{C}\hat{P}_c\mathcal{C}^T\right) &= 4\sigma_p\,\text{trace}\left(C_1\Sigma^{-1/2}C_1^T\right)+2\sigma_p^{1/2}\,\text{trace}\left(C_2C_2^T\right),\\ &= 4\sigma_p\sum_{i=1}^{p}\frac{\alpha_i}{\sigma_i}-2\alpha_p,\end{aligned}$$

where matrix $\hat{P}_c$ is the solution to the Lyapunov equation (98). The controllability Grammian $\mathcal{P}_c$ associated with the error system $\mathcal{E}_p(s)$ is given by

$$\mathcal{A}\mathcal{P}_c+\mathcal{P}_c\mathcal{A}^T+\mathcal{B}\mathcal{B}=0.$$

Notice that since $\hat{A}=\mathcal{A}$ and $\hat{B}\hat{B}^T=\mathcal{B}\mathcal{B}^T+\bar{B}\bar{B}^T\geq\mathcal{B}\mathcal{B}^T$, which implies that $\mathcal{P}_c\leq\hat{P}_c$. Therefore,

$$\|\mathcal{E}_p(s)\|_2^2=\text{trace}\left(\mathcal{C}\mathcal{P}_c\mathcal{C}^T\right)\leq\text{trace}\left(\mathcal{C}\hat{P}_c\mathcal{C}^T\right)=4\sigma_p\sum_{j=1}^{p}\frac{\alpha_j}{\sigma_j}-2\alpha_p.$$

Taking successive one step iterations one obtains that

$$\|\mathcal{E}_i(s)\|_2=\|\mathcal{G}_r^{i+1}(s)-\mathcal{G}_r^i(s)\|_2\leq 2\left(\sigma_i\sum_{j=1}^{i}\frac{\alpha_j}{\sigma_j}-\alpha_i\right)^{1/2}.$$

The bound on the fully reduced order model is then obtained through

$$\|\mathcal{G}(s)-\mathcal{G}_r(s)\|_2=\|\sum_{i=r+1}^{p}\mathcal{E}_i(s)\|_2\leq\sum_{i=r+1}^{p}\|\mathcal{E}_i(s)\|_2\leq 2\sum_{i=r+1}^{p}\left(\sigma_i\sum_{j=1}^{i}\frac{\alpha_j}{\sigma_j}-\alpha_i\right)^{1/2},$$

which completes this proof. □

The above theorem provides upperbounds for the norms of the model reduction error. The bound on the H_∞ norm given at item *b*) is sufficiently simple to be used with Algorithm 4. In fact, Algorithm 4 can be used to generate a reduced order model that minimize that bound by simply ordering the diagonal entries of Σ in (80) in ascending order (*standard balanced truncation*). This guarantee that Σ_2 will contain the largest $p-r+1$ smallest singular values, which minimizes (83). The bound on the H_2 norm given in item *c*) seems to be somewhat more involved. In fact, the minimization of (84) can not be achieved by simply ordering the singular values in ascending or descending order. A simpler alternative seems to order the singular values of Σ in descending order of Λ (*balanced truncation with component cost analysis*). Notice that balanced coordinates are also cost decoupled coordinates and this choice reflects the desire to match the output covariance as stated in Section 4.3. Notice that regardless of the singular value ordering, the bounds are still guaranteed to hold. Examples of both choices will be given in Section 5.

4.5. Singular Perturbation Truncation

In this section it is assumed that the original system has been transformed into the form

$$\begin{pmatrix} \dot{z_1} \\ \varepsilon \dot{z_2} \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \tag{99}$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + Du, \tag{100}$$

where $\varepsilon > 0$ is a sufficiently small scalar. In this form, the states z_1 and z_2 represent, respectively, "slow" and "fast" modes of the system. That structure may arise naturally in several systems. For instance, in an electrical machine, the mechanical parts generate the slow modes while the electric components are associated with the fast modes. Such systems are called *singularly perturbed systems*.

In singularly perturbed systems, the transient responses of the fast modes usually attain their steady-state in very small time, as compared with the time response of the slow modes. In this sense, the slow modes "dominate" the response of the system and, in some applications, the fast modes can be simply replaced by its steady-state solution. That is, the fast states z_2 can be simply replaced by $\tilde{z}_2(t) = \lim_{t\to\infty} z_2(t)$ whenever that limit is finite. When A_{22} is nonsingular, this quantity can be computed by solving the second equation in (99) by taking $\varepsilon \dot{z}_2 \to 0$, that is

$$A_{21}z_1 + A_{22}\tilde{z_2} + B_2 u = 0 \quad \Rightarrow \tilde{z}_2 = -A_{22}^{-1}\left(A_{21}z_1 + B_2 u\right). \tag{101}$$

The substitution of z_2 by $\tilde{z}_2$ in (99–100) provides the reduced order model

$$\dot{z_1} = A_r z_1 + B_r u, \tag{102}$$

$$y = C_r z_1 + Du, \tag{103}$$

where

$$A_r := A_{11} - A_{12}A_{22}^{-1}A_{21}, \qquad B_r := B_1 - A_{12}A_{22}^{-1}B_2, \qquad C_r := C_1 - C_2 A_{22}^{-1}A_{21}. \tag{104}$$

5. Tutorial Examples

5.1. Example 1

The use of the model reduction procedures discussed so far will be illustrated by two examples. The first example is the third order stable SISO system

$$\dot{x} = \begin{bmatrix} -1 & 3 & 0 \\ -1 & -1 & 1 \\ 4 & -5 & -4 \end{bmatrix} x + \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix} u \tag{105}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x. \tag{106}$$

Model	$\mathcal{G}_r(s)$	$\|\mathcal{G}_r(s)\|_2$	$\|\mathcal{G}_r(s)\|_\infty$	$\|\mathcal{G}(s)-\mathcal{G}_r(s)\|_2$	$\|\mathcal{G}(s)-\mathcal{G}_r(s)\|_\infty$
Full	see (107)	1.41	2.00	$\star$	$\star$
#1, #2	$\dfrac{-2}{s+1}$	1.41	2.00	2.34	4.00
#3	$\dfrac{0.69}{s+0.35}$	0.83	2.00	1.11	0.78
#4	$\dfrac{0.58}{s+0.21}$	0.89	2.72	1.12	0.77
#5	$\dfrac{-2.85}{s+3.41}$	1.09	0.84	1.61	2.84
#6	$\dfrac{-2.02(s-2.09)}{(s+0.78)(s+2.51)}$	1.41	2.15	0.24	0.23
#7	$\dfrac{-2.54(s-2.24)}{(s+0.71)(s+4.03)}$	1.37	2.00	0.20	0.14
#8	$\dfrac{-2(s-2.12)}{(s+1)(s+2.12)}$	1.41	2.00	0.22	0.21
#9	$\dfrac{-2.27(s-2.16)}{(s+0.98)(s+2.64)}$	1.41	1.89	1.15	0.12

Table 1: Example 1: model comparison

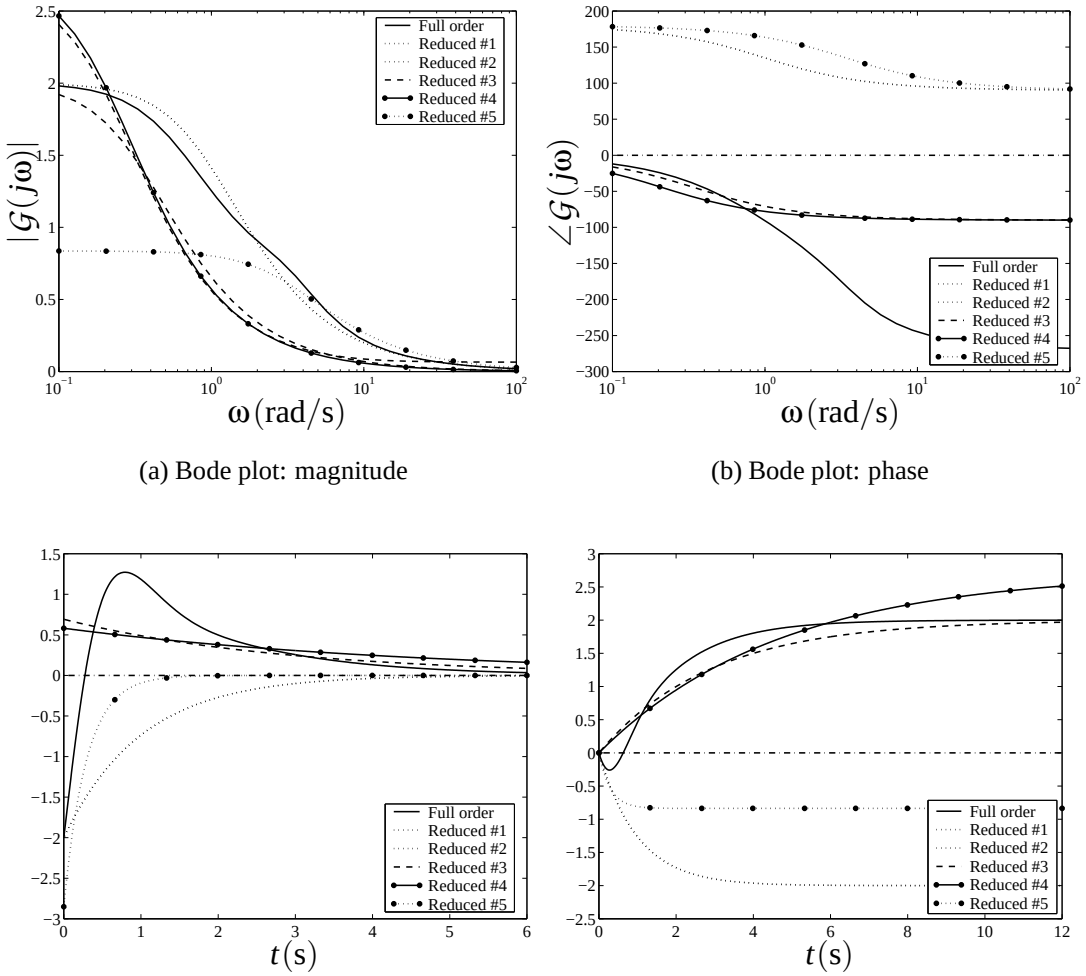
This system is strictly proper and its transfer function is given by

$$H_{uy}(s) = \frac{-2(s-2.16)(s+4.16)}{(s+0.67)(s^2+5.33s+13.43)}. \tag{107}$$

Notice that this transfer function has a zero on the right half of the complex plane, hence it is *nonminimum phase*.

First, five different reduced order models of order one have been produced using the techniques introduced in this chapter. The bode diagrams of the full order model (105–106) and the reduced order models is depicted in Figures 5(a) and 5(b). The impulse response and the step response are given respectively in 5(c) and 5(d). All the obtained models and the H_2 and H_∞ norms of the model error are given in Table 1. Then, five different reduced order models of order two have been generated. The bode diagrams of the full order model and the obtained reduced order models are in Figures 6(a) and 6(b). The impulse response and the step response are in Figures 6(c) and 5(d). A comparison of the models is given in Table 1. Some comments on the generation of the reduced order models and their performance follow.

Reduced order model #1 (component cost analysis) The reduced order model #1 has been obtained using Algorithm 3 with $n_r = 1$. Notice that CC^T has rank one ($\Lambda = \mathrm{diag}\,(2,0,0)$) and, as expected, the reduced order model matches exactly the H_2 norm of (105–106). Notice that this does



(a) Bode plot: magnitude

(b) Bode plot: phase

(c) Impulse response

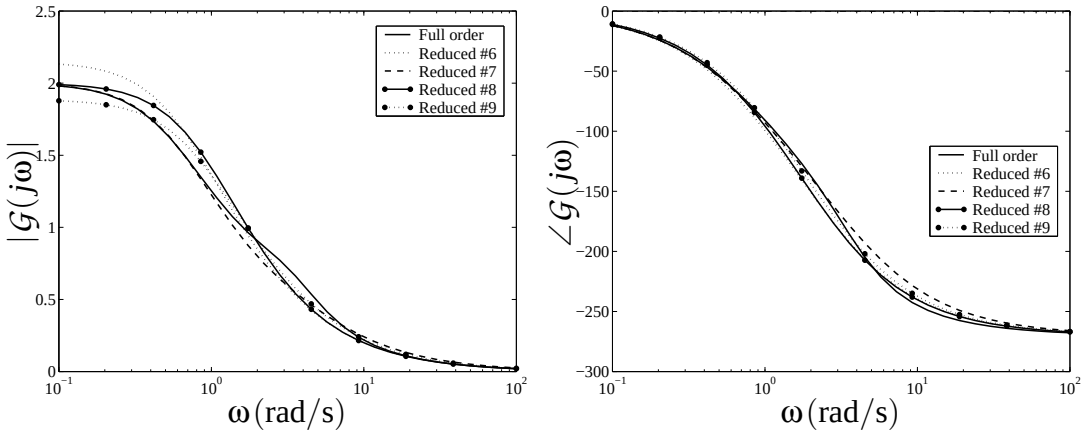
(d) Step response

Figure 5: Example 1: reduced order models of order one

not guarantee any property on the H_2 norm of the model reduction error. Indeed, Table 1 reveals that this reduced order model is outperformed by all other models.

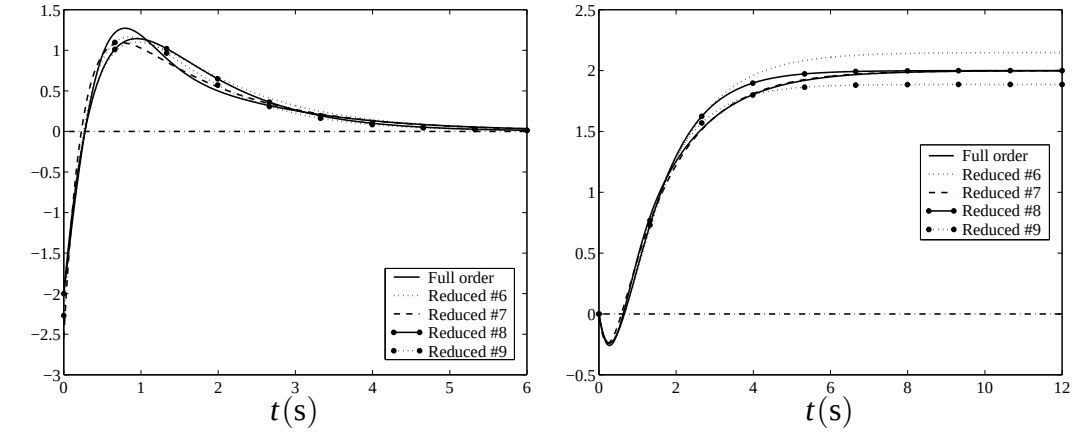
Reduced order model #2 and #3: (moment match) The reduced order model #2 has been obtained using Algorithm 2 with $q = 1, p = 0$. That is, only the first high-frequency moment (Markov parameter) of system (105–106) has been matched. As the system is nonminimum phase, the obtained reduced order model has phase 180 deg at $\omega = 0$. In this example, the reduced model #1, designed by component cost analysis, is coincidently the same as model #2. Notice that in the design of the reduced order model #1 only the first high-frequency power moment (and not the first Markov parameter) is guaranteed to be match, and the two disctinct methods should provide different models in more complex situations.

The reduced order model #3 has been obtained using Algorithm 2 with $q = 0, p = 1$. The first low-frequency moment has been matched, which guarantees that the steady-state value of the step response



(a) Bode plot: magnitude

(b) Bode plot: phase



(c) Impulse response

(d) Step response

Figure 6: Example 1: reduced order models of order two

will be matched (see figure 5(d)).

Reduced order model #4 and #5: (balanced truncation) Transforming the system (105–106) into balanced coordinates produces

$$P_{\mathrm{o}}^{1/2} = P_{\mathrm{c}}^{1/2} = \Sigma := \mathrm{diag}\,(\sigma_i) = \mathrm{diag}\,(1.8541, 0.1750, 0.0032)\,.$$

The reduced order model #4 has been obtained using Algorithm 4 with $n_r = 1$ and the singular values ordered in decreasing order of Σ (standard balanced truncation). Notice that all entries are distinct and that σ_1 is one order larger than σ_2 and three orders larger than σ_3, which suggests that a model of order one might be used to approximate (105–106). In balanced coordinates, the component cost analysis produces

$$\Lambda := \mathrm{diag}\,(\alpha_i) = \mathrm{diag}\,(0.7921, 1.1926, 0.0153)\,,$$

which suggests that the second singular value should be kept, if the H_2 norm is to be matched. The reduced order model #5 has been obtained using Algorithm 4 with $n_r = p = 1$ and the singular values

ordered in decreasing order of Λ (balanced truncation with component cost analysis).

Reduced order model #6: (component cost analysis) In order to generate model #6 the output of system (105–106) has been augmented to

$$\tilde{y} = \begin{bmatrix} y^T & x^T(CA^{-1})^T & x^T(CA^{-2})^T \end{bmatrix}^T$$

and Algorithm 3 has been used with $n_r = 1$. As commented in Section 4.3, the purpose of augmenting the output is to increase the "meaning" of the output covariance to be matched. In this case, the augmented outputs introduce terms that are similar to low-frequency power moments in the output covariance. As the number of outputs have increase the obtained

$$\Lambda := \operatorname{diag}(\alpha_i) = \operatorname{diag}(20.2391, 1.4338, 0.0034)$$

has now three distinct diagonal entries. The number of outputs in the reduced order model produced by Algorithm 3 now reflects the dimension of the augmented output and a model that is appropriate for comparison is obtained by keeping only the first output.

Reduced order model #7 and #8: (moment match) The reduced order model #7 has been obtained using Algorithm 2 with $q = 0, p = 2$. That is, the two first low-frequency moments of system (105–106) have been matched. The reduced order model #8 has been obtained using Algorithm 2 with $q = 1, p = 1$. The first low-frequency moment and the first Markov parameter have been matched. As both models match the first low-frequency moment, the steady-state of the step response should be exactly reproduced (see Figure 6(d)). Models #7 and #8 are very similar and present excellent performance. Notice that the norms of the model reduction error is also very small even though the moment matching does not guarantee any norm bounds. Also notice that model #7 captures the nonminimum phase characteristics of the model even without explicitly being asked to match the first high frequency moment.

Reduced order model #9: (balanced truncation) For a second order system, the ordering of the singular values coincides for both matrices Σ and Λ (see the generation of models #4 and #5) and Algorithm 4 with $n_r = p = 2$ produces a unique system, the reduced order model #9. The obtained model performs very well and capture the nonminimum phase characteristics. There remains a small error in the steady state response to a step input.

5.2. Example 2

The second example is the four mass mechanical system with springs and dampers depicted in Figure 7. The system has a vector second order representation

$$M\ddot{q} + G\dot{q} + Kq = Du, \tag{108}$$

$$y = Pq + Q\dot{q}, \tag{109}$$

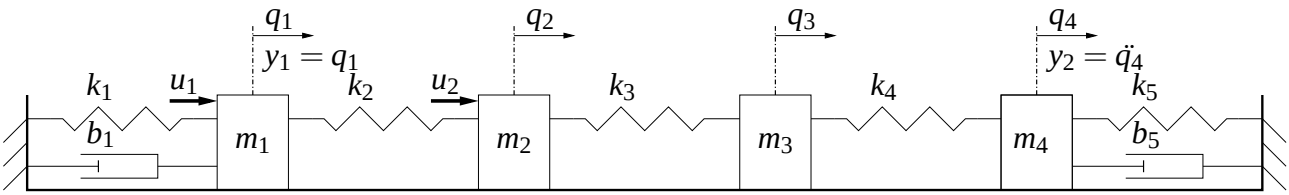


Figure 7: Four mass mechanical system

Model	$\|\mathcal{G}_r(s)\|_2$	$\|\mathcal{G}_r(s)\|_\infty$	$\|\mathcal{G}(s)-\mathcal{G}_r(s)\|_2$	$\|\mathcal{G}(s)-\mathcal{G}_r(s)\|_\infty$
Full	0.65	1.23	$\star$	$\star$
#1	0.59	1.12	0.34	0.55
#2	0.65	1.37	0.39	0.75
#3	0.58	1.44	0.36	0.54
#4	0.61	1.38	0.53	0.85

Table 2: Example 2: model comparison

where

$$M = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix}, \quad G = \begin{bmatrix} b_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_5 \end{bmatrix}, \quad K = \begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4+k_5 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^T, \quad Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_5/m_4 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & k_4/m_4 & -(k_4+k_5)/m_4 \end{bmatrix},$$

and

$$m_1 = m_2 = m_4 = 1, \quad m_3 = 2, \quad k_1 = k_3 = k_4, \quad k_2 = 2, \quad k_5 = 4, \quad b_1 = 2, \quad b_5 = 1$$

Before applying the model reduction procedures this system with four modes should be converted into state-space form (with eight states) using, for instance,

$$\dot{x} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}G \end{bmatrix} x + \begin{bmatrix} 0 \\ M^{-1}D \end{bmatrix} u, \quad x = \begin{pmatrix} q \\ \dot{q} \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad y = \begin{pmatrix} q_1 \\ \ddot{q}_2 \end{pmatrix},$$
$$y = \begin{bmatrix} P & Q \end{bmatrix} x.$$

This is a MIMO system with 2 inputs and 2 outputs. The singular values and the component costs obtained in balanced coordinates are

$$\Sigma := \operatorname{diag}(\sigma_i) = \operatorname{diag}(0.7373, 0.3060, 0.1722, 0.1394, 0.0735, 0.0149, 0.0015, 0.0009),$$
$$\Lambda := \operatorname{diag}(\alpha_i) = \operatorname{diag}(0.1927, 0.0745, 0.0117, 0.0632, 0.0474, 0.0318, 0.0026, 0.0004)$$

which suggests that a model of order four can be possibly used to approximate this eight order system. Six different reduced order models have been computed according to the following guidelines. A brief comparison chart is given in Table 2.

Reduced order model #1 and #2: (moment match) The reduced order model #1 has been obtained by matching the first two low frequency moments of the original system. The reduced order model #2 has been calculated to match the first low frequency and the first high frequency moments. Both models have been calculated using Algorithm 2. As the full order system has two outputs fourth order systems are needed to match two moments (two degrees per moment). According to Table 2 the performance of model #1 is better than that of model #2. This fact is justified since the original system has a more prominent response on low frequency ranges, and matching low frequency information (moments) seems to capture better these features. The fact that the system is minimum phase also reduced the importance of matching high frequency moments.

Reduced order model #3 and #4: (balanced truncation) Models #3 and #4 have been obtained by balanced truncation using Algorithm 4 with $n_r = p = 4$. In model #3 the singular values have been ordered in descending order (standard balanced truncation) while in model #4 the singular values have been ordered in descending order of the component cost matrix Λ (balanced truncation with component cost analysis).

6. Concluding Remarks

We have tried to show that the bottleneck to good system design is the lack of good system modeling theories. Indeed, we have shown that good component models do not imply good system models. The controller and the model are compatible, or not, as a pair, and neither the controller nor the model have any significance in isolation of each other. All the investment we've made into modeling the minutia of component technologies may have little value in the discipline of System modeling. Whether you are given a controller and are designing a model, or given a model designing a controller, it's an iterative process. Obviously control design should occur during, and not after modeling. We should abandon the classical approach of designing and modeling the plant before control design. This will require more attention in the future to good systems design procedures, integrating the disciplines of modeling and control, rather than merely employing these disciplines in a sequential manner, as in the current practice.

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