Open-loop shape control of stable unit tensegrity structures

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ABSTRACT

Tensegrity deployment is considered herein as a tracking control problem. Therefore, the required trajectories should be feasible for a given structure. For tensegrity structures, this means that in every desired configuration, the structure has to satisfy tensegrity conditions, which require strings to be in tension, and the structure to be stable. To define an open-loop deployment control law, geometry parameterization of those configurations and corresponding rest lengths of string elements guaranteeing equilibrium are defined first. By slowly varying desired geometry, an open-loop string rest length control is defined. This makes the structure track trajectories defined by the time dependent geometry parameters. Deployment of plates made of stable symmetric shell class tensegrity units is given as an example.
Keywords: Tensegrity, Deployment, Control, Rest length

1 INTRODUCTION

A tensegrity structure is a prestressable stable dynamical truss-like system made of axially loaded bar and string elements. Only admissible connections between the elements are ball joints. String elements of a tensegrity structure are modeled as elements capable of transmitting tension load only. Unlike regular trusses, tensegrities allow a smaller set of admissible topologies than the set of topologies that yield mechanisms. Defining a set of topologies that do not have mechanisms is a main tensegrity design issue.

Tensegrity structures as an art form were first introduced in 1948 by Keneth Snelson [1]. R.B. Fuller [2] was the first one to recognize their engineering values. Over the course of past 50 years, tensegrity structures were analyzed mostly in a descriptive meaner. All the designs were usually obtained by ingenuity of their authors. Pelegrino [3, 4], Motro [5], Hanaor [6], were among the first to recognize the importance of developing systematic design techniques to solve statics problems for the tensegrity structures. The work of Skelton and Sultan [7–11] introduced the concept of controlled tensegrity structures.

2 SOME PRELIMINARY NOTIONS

Based on element connectivity property and level of symmetry in a designed tensegrity structure, all design concepts can be divided in the following three categories:

- *Stable unit tensegrity design*. This concept assumes that the tensegrity structure is built of units which themselves are stable tensegrities.
- *Unstable unit tensegrity design*. This approach involves a quasi-periodic connectivity scheme. Tensegrity designs are characterized by periodicity in the way the elements are connected, but this pattern is usually violated on the boundaries of the structure, by adding extra elements, to stabilize the whole structure.
- *General tensegrity design*. This concept allows all the elements comprising a tensegrity to be connected in an arbitrary way.

Since the first two concepts are special cases of the general tensegrity design problem, its formulation is given first. Once a maximum set of allowed element connections of a tensegrity structure and its associated oriented graph have been adopted, corresponding connectivity information are written in a form of member-node incidence matrix, $M \in \mathbf{R}^{3n_e \times 3n_n}$, where n_e, n_n are number of elements and nodes respectively. Matrix M is a sparse block matrix whose i, j block is I_3 or $-I_3$ if the element i ends at or emanates from the j^{th} node, otherwise it is 0_3. Element force t_i of a prestressed element i of a tensegrity structure can be expressed as a product of an element vector g_i, and a scaling factors λ_i, called a force coefficient. Vector $g \in \mathbf{R}^{3n_e}$, formed by stacking up all the element vectors g_i, is a linear mapping of a nodal position vector, $p \in \mathbf{R}^{3n_n}$,

$$g = \begin{bmatrix} g_s \\ g_b \end{bmatrix} = Mp, \quad M = \begin{bmatrix} -S^T \\ B^T \end{bmatrix}, \quad S \in \mathbf{R}^{3n_n \times 3n_s}.$$

If a structure admits symmetry Φ, then elements in n_{g_e} groups of equivalent elements in the structure, share the same forces. This enables a full vector of force coefficients $\lambda \in \mathbf{R}^{n_e}$, formed by stacking up force coefficients λ_i, to be expressed as a linear mapping $Q \in R^{n_e \times n_{eg}}$ of a set of independent force coefficients $\underline{\lambda}$. Q is defined as an element-group of equivalent elements, incidence matrix. Structural symmetry also allows reduction of number of force balance equations, since for equivalent nodes, they are related through the same symmetry of the structure. This reduction can be accomplished by multiplying the full set of force balance equations for each of nodes by a matrix $D \in R^{3n_{g_n} \times 3n_n}$, where n_{g_n} is number of groups of the equivalent nodes. D is defined as a group of equivalent nodes-node, incidence matrix of a proper size. Node position symmetry enables parametrization of the full vector p to reduce number of variables appearing in the problem, which is done by writing $p = p(\alpha, \beta, \gamma, \ldots)$, where $\alpha, \beta, \gamma, \ldots$ are geometry parameters defining shape of a structure. To define desired shape of a tensegrity structure, shape constrains $\varphi(\alpha, \beta, \gamma, \ldots) = 0$ are added. Finally, a general tensegrity design problem in design variables $\alpha, \beta, \gamma, \ldots, \underline{\lambda}$ can be written as,

$$\begin{aligned} & DC\tilde{g}Q\underline{\lambda} = 0, \quad \underline{\lambda} \geq 0, \quad i \in I_s, \quad C = \begin{bmatrix} S & B \end{bmatrix}, \\ & p = p(\alpha, \beta, \gamma, \ldots), \quad g = \begin{bmatrix} -S^T \\ B^T \end{bmatrix} p, \\ & \varphi(\alpha, \beta, \gamma, \ldots) = 0 \end{aligned} \tag{1.1}$$

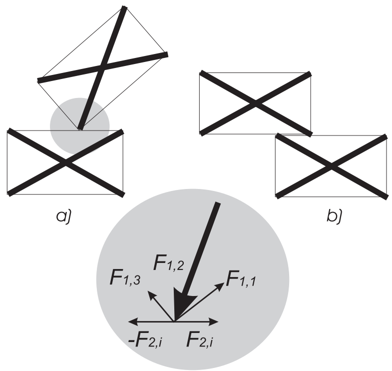

Figure 1 Connection of two tensegrity units

where I_s is a set of indices of string elements. Linear operator $\tilde{\cdot}$ is defined as:

$$\mathbf{R}^{3n} \to \mathbf{R}^{3n\times n}, \quad \tilde{x} = blockdiag\{x_i\}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{bmatrix}, x_i = \begin{bmatrix} x_{i_1} \\ \cdot \\ x_{i_3} \end{bmatrix}. \tag{1.2}$$

This paper deals with stable unit tensegrities. It is much easier to design and analyze such a structure. Tensegrity designs of this class are obtained without directly having to solve the global equations (1.1).

2.1 Equilibrium analysis of superposition of two tensegrities that are in equilibrium

It can be shown that equations (1.1) have particular structure when corresponding tensegrity is of the stable unit class. Equilibrium conditions can be divided in decoupled blocks of equations, so that each of them represent equilibrium condition for each separate unit. Several definitions and rules will help establish the procedure to do this. These rules provide a technique for connecting the tensegrity units in a bigger structure whose tensegrity conditions can easily be computed. This technique will be called *structure superposition*. Although, for some readers, these claims may seam obvious, they are useful to define a systematic technique for the stable unit tensegrity design.

Definition 1 *Node i of a structure* 1 *is said to be an* **attached node** *to a structure* 2 *if:*

(i) It is placed on the structure 2 *without changing direction of its elements.*

(ii) If it is placed on the span of an element j of the structure 2*, this element is divided in two elements connecting the node i to nodes defining the element j.*

(iii) Element forces of these new elements are equal to element force of the element j.

Definition 2 *An element g, connecting nodes i and j, is said to be a* **superposition of two elements** *k and l, defined by the same nodes i and j, if the overlapping elements k and l are*

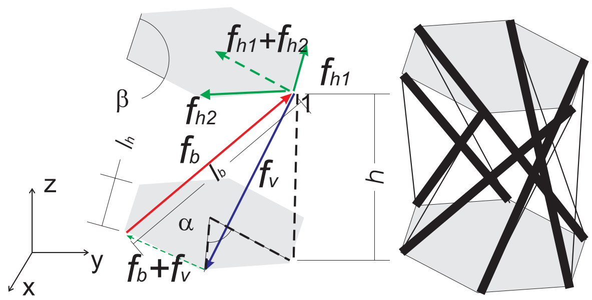

Figure 2 Geometry and element forces of the n-bar one stage shell class tensegrity

substituted by the element g, whose force is the sum of forces of the elements k and l. Elements k and l are called **superposed elements***.*

Definition 3 *A structure is said to be a* **superposition of two structures***, if:*

(i) n nodes of structure 1 *are attached to structure* 2 *and m nodes of structure* 2 *are attached to structure* 1*.*

(ii) If overlapping elements are generated, they are replaced by their superposition.

Structures 1 *and* 2 *are called* **superposed structures***.*

The following fact defines the main property of structures formed by superposition of equilibrium structures. It is given without a proof.

Fact 1 *A structure that is a superposition of two equilibrium structures is in equilibrium.*

3 TENSEGRITY PLATE DEPLOYMENT

3.1 Equilibrium analysis of a one stage n-bar shell class tensegrity

The geometry of such a structure is depicted in *Fig. 2*.

Theorem 1 *All symmetric one stage n-bar shell class tensegrities are in equilibrium if the twist angle between the top and bottom polygons is $\alpha = \frac{\pi}{2} - \frac{\pi}{n}$. Force coefficients λ_v of all the vertical elements are equal. Force coefficients λ_h, of horizontal members, are computed as:*

$$\lambda_h = \frac{\lambda_v}{2\cos\alpha} \tag{1.3}$$

where n is number of bars in the unit.

Proof: Due to the symmetry of the structure, it is sufficient to compute balance of the forces at only one of the nodes. Balance of the forces at z direction is,

$$f_{bz} - f_{vz} = 0, \tag{1.4}$$

which can be expressed in terms of the force coefficients as,

$$\lambda_{v_b} h - \lambda_{v_s} h = 0, \tag{1.5}$$

where λ_{v_b} and λ_{v_s} are force coefficients of bars end vertical strings respectively. Rewriting (1.5),

$$(\lambda_{v_b} - \lambda_{v_s})h = 0, \tag{1.6}$$

from (1.6) $\lambda_{v_b} = \lambda_{v_s}$.

To prove that $\alpha = \frac{\pi}{2} - \frac{\pi}{n}$ recall that if λ is a set of the force coefficients of the elements of the structure then $a\lambda$, $a > 0$ is also a set of the force coefficients satisfying equilibrium. Then, without lost of generality, a can be chosen such that $\lambda_{v_s} = \lambda_{v_b} = 1$. For this choice of the force coefficients, $\vec{f}_v + \vec{f}_b$ takes the direction of the horizontal string connecting their ends opposite from the node 1 as depicted in the *Fig.2*. $\vec{f}_{h1} + \vec{f}_{h2} = -(\vec{f}_v + \vec{f}_b)$ must be satisfied for equilibrium at the node 1. Due to the symmetry of the forces, $\vec{f}_{h1} + \vec{f}_{h2}$ takes the direction of the bisector of the angle $\beta = \frac{(n-2)\pi}{n}$ between them, that is an angle between sides of a n-sided polygon. In order for this direction to match direction of $\vec{f}_v + \vec{f}_b$, the twist angle α has to be $\frac{1}{2}$ of the angle β, $\alpha = \frac{1}{2}(\frac{(n-2)\pi}{n}) = \frac{\pi}{2} - \frac{\pi}{n}$.

From *Fig.2* it is clear that the magnitude of the resulting force of the horizontal strings is:

$$f_r = \|\vec{f}_{h1} + \vec{f}_{h2}\| = 2 f_{h1} \cos\alpha = 2k_h l_h \cos\alpha \tag{1.7}$$

Since the magnitude of the resultant force $\vec{f}_r$ need to be the same as the length of the horizontal string l_h,

$$2\lambda_h \cos\alpha = 1, \tag{1.8}$$

and finally,

$$\lambda_h = \frac{1}{2\cos\alpha}. \tag{1.9}$$

The computed λ_h corresponds to the choice $\lambda_v = 1$. For any other choice of λ_v, λ_h is computed as follows:

$$\lambda_h = \frac{\lambda_v}{2\cos\alpha}. \tag{1.10}$$

□

3.2 Tensegrity plate equilibrium and string rest length shape control law

A class of tensegrity plates can be derived by superposition of one stage shell class tensegrity units. *Fig.3* shows how to superpose units in the cases where three-bar, four-bar and six-bar units are used. If all the units have common geometry, then the geometry of the formed plate is defined by parameters of the units and parameter $\gamma = \frac{l_4}{l_2}$ as shown in the *Fig.4*. Note that the parameter γ defines the relative position of the units within the palate. In order for two units to be superposed, it can be shown that for three-bar and four-bar unit plates this parameter must have fixed values $\gamma = 2 - \sqrt{3}$ and $\gamma = 2 - \sqrt{2}$ respectively. For the six-bar unit plate this parameter is $0 < \gamma < 1$. In the case that $\lambda_v > 0$ is equal for both of the units, forces satisfying

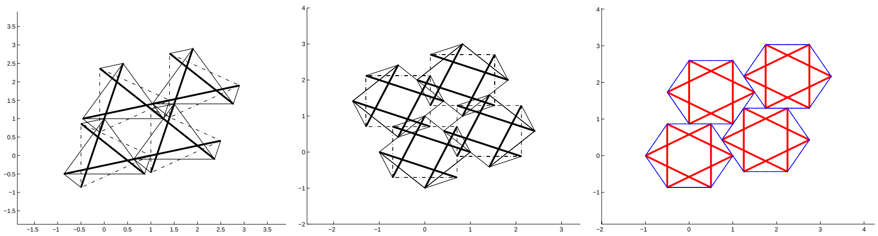

Figure 3 Tensegrity plate made connecting three-bar, four-bar and six-bar units.

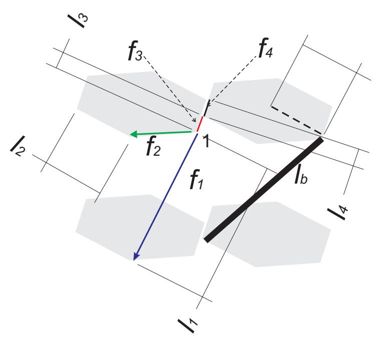

Figure 4 Superposition of two n-bar units to build a plate.

equilibrium depicted in *Fig. 4* are:

$$f_1 = \lambda_v l_1, \quad f_2 = \lambda_h l_2 = \lambda_v \frac{1}{2\cos\alpha} l_2, \quad f_3 = f_2,$$
$$f_4 = 2f_2,\ n = 4,6, \quad f_4 = f_2,\ n = 3 \tag{1.11}$$

Since geometry of a n-bar unit in stable equilibrium is uniquely defined by two parameters, l_b and r, as shown in *Fig. 4*, and geometry of a plate that is formed by superposition of the units by the additional parameter γ, lengths of the elements of a tensegrity plate are computed as follows:

$$l_1 = \sqrt{l_b^2 + 2r^2(\cos(\alpha + \frac{2\pi}{n}) - \cos\alpha)}, \quad l_2 = 2r\sin(\frac{\pi}{n}),$$
$$l_4 = \gamma l_2, \quad l_3 = (1-\gamma) l_2, \quad 0 < \gamma < 1. \tag{1.12}$$

Let l_{0_2} of the string 2 be defined as a fraction of its total length l_2,

$$l_{0_2} = \delta l_2 = \delta 2r \sin(\frac{\pi}{n}), \quad 0 < \delta < 1. \tag{1.13}$$

Let the material of the structure be linear elastic:

$$f_i = \frac{A_i E_i}{l_{0_i}}(l_i - l_{0_i}), \tag{1.14}$$

and area of cross section and Young's modulus A_i and E_i be the same for all elements. Then (1.11)-(1.14) and the fact that $\alpha = \frac{\pi}{2} - \frac{\pi}{n}$, give a complete parameterization of the remaining rest lengths in terms of the free geometry parameters l_b, r, γ,

$$\begin{aligned} l_{01} &= \frac{r\sqrt{l_b^2 - 4r^2\sin(\frac{\pi}{n})}\, l_{02}}{r l_{02} + \sqrt{l_b^2 - 4r^2\sin(\frac{\pi}{n})}(2r\sin\frac{\pi}{n} - l_{02})}, \\ l_{03} &= (1-\gamma) l_{02}, \\ l_{04} &= \frac{2r\gamma l_{02}\sin(\frac{\pi}{n})}{4r\sin(\frac{\pi}{n}) - l_{02}}, \; n = 4,6, \quad l_{04} = \gamma l_{0_2}, \; n = 3. \end{aligned} \tag{1.15}$$

Defining sufficiently slowly varying functions of desired geometry parameters change, $r = r(t)$ and $\gamma = \gamma(t)$ in the six-bar unit case, and possibly $l_b = l_b(t)$, and substituting these function in (1.13),and (1.15), an open-loop string rest length control law for the tensegrity to track desired trajectories is defined.

3.3 Simulation results

The open-loop control strategy defined was applied on a full nonlinear model of a six-bar unit tensegrity plate. Size of the plate was 6×6 units. Deployment shown involved a desired radius and overlap parameter change functions as follows:

$$\begin{aligned} r(t) &= 0.2 + 0.04t, && 0 < t < 20, \\ r(t) &= 1, && 20 < t < 30 \end{aligned} \tag{1.16}$$

$$\begin{aligned} \gamma(t) &= 0.2 + 0.02t, && 0 < t < 20, \\ \gamma(t) &= 0.6, && 20 < t < 30 \end{aligned} \tag{1.17}$$

The pre-stress parameter δ was kept constant. The final unit polygon radius was 1 which yields a ratio between the areas of the plate in the final and initial configuration of approximately 25. Simulation results are depicted in *Fig.5*.

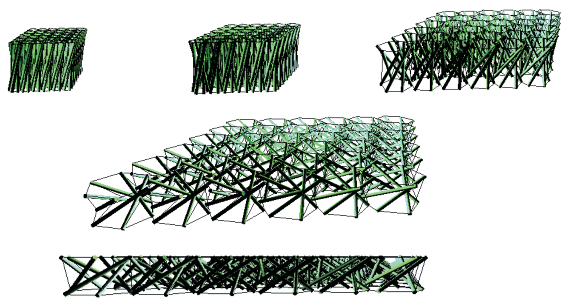

Figure 5 Deployment of a six-bar unit plate.

Setting up the rest lengths of the strings lying on the top surface of the plate to be a fraction of the rest lengths of the corresponding bottom strings, while keeping the rest length change law for all other strings unaltered, parabolic like structure is obtained. The ratio $q(t) = \frac{l_{o_{top}}(t)}{l_{o_{bottom}}(t)}$ between rest lengths of the top and bottom strings is as follows:

$$\begin{aligned} q(t) &= 1, & 0 < t < 20, \\ q(t) &= 1 - 0.02(t-20), & 20 < t < 25, \\ q(t) &= 0.9, & 25 < t < 30. \end{aligned} \tag{1.18}$$

Deployment simulation of the parabolic structure is depicted in *Fig.6*. Deviation of the nodal positions from the quadratic parabola shape is shown in *Fig.7*

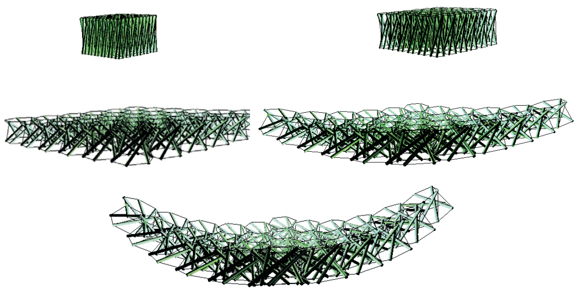

Figure 6 Deployment of a parabolic tensegrity structure.

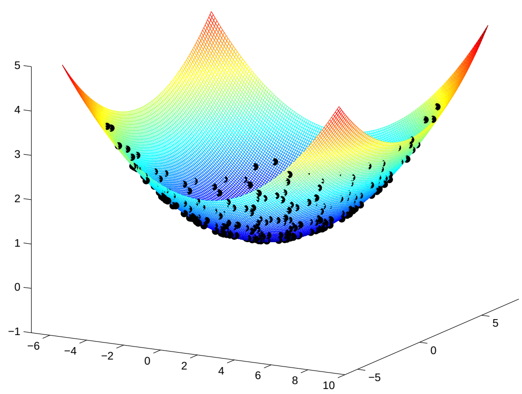

Figure 7 Nodal position parabolic surface fitting.

4 CONCLUSIONS

In the preceding analysis a decentralized open-loop control law for stable unit tensegrity shape control is defined. The introduced controlled strategy can be generalized to any tensegrity whose stable equilibrium configurations are parametrizeable in terms of geometry parameters. Simple parabolic example shows that tensegrity plates can be successfully used to obtain different desired curved shapes. This makes stable unit tensegrity plates promising technologies for deployable space structures. Possible applications are deployable antennae, mirrors and dome structures. Future work will address these problems in more details.

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