

OPTIMAL TOPOLOGY AND GEOMETRY FOR CONTROLLABLE TENSEGRITY SYSTEMS

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Abstract: This paper demonstrates a procedure to design an optimal mass to stiffness ratio tensegrity structure. Starting from an initial layout of the structure that defines an allowed set of element connections, the procedure defines positions of the nodal points of the structure, volumes of the elements and their rest lengths yielding a tensegrity structure having smaller compliance for a given load applied than an initial design. To satisfy design requirements strength constraint for all the elements of the structure, buckling constraint for bar elements as well as constraint on geometry of the structure are imposed yielding a nonconvex nonlinear constrained optimization problem. Structural static response is computed using complete nonlinear large displacement model. Examples showing optimal layout of a 2D and 3D structure are shown.

Keywords: Tensegrity Structure, Optimal Stiffness, Design Constraints, Nonlinear Program

1. INTRODUCTION

A tensegrity structure is a prestressable stable dynamical truss-like system made of axially loaded elements. What differentiates them from regular truss structures is that all tensile elements are strings capable of transmitting load in only one direction. Unlike regular trusses a set of admissible topologies is much smaller than the set of topologies that yield a structure containing mechanisms.

Tensegrity structures as an art form were first introduced by Snelson (1965). Fuller (1962) was the first one to recognize their engineering values. Over the course of the 50 years from the moment of their first creation, tensegrity structures were analyzed mostly in a descriptive manner. Experimental and geometrical analysis techniques prevailed. No systematic design and analysis procedures were defined. All the designs were usually obtained by ingenuity of their authors. The importance of developing systematic design techniques for the tensegrity structures is recognized

in works of several authors (Pellegrino and Calladine, 1985), (Hanaor, 1992), (Skelton *et al.*, 2001).

Some of the advantages of tensegrity structures are:

- (1) all elements are loaded axially only, this type of load can be more efficiently carried than bending loads,
- (2) the choice of material can be specialized on axial loads, and split further in material optimized for compressive and tensile stresses and strains. The same holds for element geometry.

One of the properties that sets tensegrity structures apart from most of the structures used in practice is that they are very suitable for shape control. By controlling rest lengths of the string elements it is possible to control a desired shape of the structure. Tensegrities can easily be stowed in a small volume transported to the desired location and deployed. This makes them applicable for different space structures, like deployable antennas, mirrors as well as deployable domes. Masic and Skelton (2001) define an open-loop control

law for shape control of stable unit tensegrity structures.

Tendons in tensegrity structures have multiple roles, they:

- rigidize and stiffen the structure
- carry structural loads,
- provide opportunities for actuation/sensing (Skelton and Adhikari, 1998).

Actuation can improve properties like stiffness or stiffness-to-mass ratio and damping, and enables shape control strategies to be used. Sensing provides information about the geometry of the structure and the deformations. Actuation can be carried out by changing the length of the tendons or the bars. This can be done in several ways, by:

- shape memory alloys that enable the tendons to shorten and lengthen by changes in temperature,
- linear or rotary motors that can shorten a tendon by hauling it, *e.g.* inside hollow bars,
- extensible bars.

Optimization of topology of structures has been studied for a long time. One of the results is the formulation of Optimality Criteria (Save *et al.*, n.d.), which has been worked out almost completely for grillages while for trusses there is also a set of conditions, but not nearly as complete as for grillages (Rozvany, n.d.). Furthermore, several approaches for numerical optimization are known, while recent approaches are, *e.g.* free material modeling (Bendsoe, 1989; Bendsoe, 1995; Sprekels and Tiba, 1998; Ben-Tal *et al.*, 1999; Sigmund, 2001), or optimization of trusses starting from a fully populated grid (Ben-Tal and Nemirovski, 1997; Jarre *et al.*, 1998). Practical purposes require to:

- incorporate constraints (nonlinear) for failure of the structure, like yield and buckling,
- tackle a wide class of geometries and boundary and loading conditions which excludes approaches using local linearization,
- stabilize the system by requiring pre-stress in the structure.

The goal of the paper is to reduce this gap in the knowledge base and to:

- outline the topology/geometry optimization, incorporating requirements for static equilibria for pre-stressed mechanical structures, both loaded and unloaded,
- show the influence of incorporating nonlinear failure constraints in the optimization, like yield and buckling,
- provide evidence that the approach yields physically relevant topologies and geometries,
- investigate the handling of requirements of installing actuating devices to control the length of the tendons, by excluding a certain range of tendon lengths.

The paper is structured as follows. First, a model for static equilibria for pre-stressed structures is outlined. Then the optimization problem is formulated, which will be compared with an LP approach based on a linearization of the problem. This is followed by an application for a planar tensegrity beam structure. A discussion with conclusions and recommendations finishes the paper.

2. FORMULATION OF THE PROBLEM

The objective of this analysis is to design a tensegrity structure that for a given mass of the material available has an optimal stiffness. In other words mass to stiffness ratio is minimized. Assuming that all the elements are made of the same material, fixing the mass available is equivalent to specifying total volume v_{total} of the material used. The optimization algorithm to a tensegrity structure whose number of nodes, strings and total number of elements available are n_n, n_s, n_{el} respectively, assigns structural parameters collected in vectors of the volumes of the elements $v \in \mathbf{R}^{n_{el}}$, rest lengths of the elements $l_0 \in \mathbf{R}^{n_{el}}$ and nodal positions $p \in \mathbf{R}^{3n_n}$. For a given vector of applied external nodal forces $f \in \mathbf{R}^{3n_n}$, this set of parameters defines a structure, whose static response, defined in the vector of nodal displacement $u \in \mathbf{R}^{3n_n}$, yields a compliance energy $\frac{1}{2}f^T u$, that is guaranteed to be improved from the the value corresponding to an initial design. Note that compliance is used as a measure of the stiffness of the structure.

2.1 Tensegrity constitutive equations

Once a maximum set of allowed element connections of a tensegrity structure and its associated oriented graph have been adopted, the corresponding connectivity information is written in a form of a member-node incidence matrix, $M \in \mathbf{R}^{3n_{el} \times 3n_n}$. Matrix M is a sparse block matrix whose i, j block is I_3 or $-I_3$ if the element i ends at or emanates from the j^{th} node, otherwise it is 0_3 . After expressing element force t_i of a prestressed element i of a tensegrity structure as a product of an element vector g_i , and a scaling factors λ_i , called a force coefficient and writing vector $g \in \mathbf{R}^{3n_{el}}$, formed by stacking up all the element vectors g_i , as a linear mapping of a nodal position vector, $p \in \mathbf{R}^{3n_n}$,

$$g = Mp, \quad M = \begin{bmatrix} -S^T \\ B^T \end{bmatrix}, \quad S \in \mathbf{R}^{3n_n \times 3n_s},$$

balance of the element forces at each of the nodes of the prestressed tensegrity structure is written as,

$$C\hat{\lambda}Mp = 0, \quad \lambda_i \geq 0, \quad C = [S \ B].$$

$\lambda \in \mathbf{R}^{n_{el}}$ is formed by stacking up force coefficients λ_i . Linear operator $(\hat{\cdot})$ is defined as:

$$\hat{\cdot} : \mathbf{R}^n \rightarrow \mathbf{R}^{3n \times 3n}, \hat{\lambda} = \text{blockdiag}\{\lambda_i I_3\}, i = 1 \dots n \quad (1)$$

Depending on the material model chosen, the relationship between force coefficients, λ and physical parameters of the structure may be different. For this analysis the linear elastic material model is used. Then force coefficients of the tensegrity structure λ and $\bar{\lambda}$, at configurations defined by the vectors of nodal positions p and $p + u$, are computed as,

$$\lambda_i = \frac{\|t_i\|_2}{l_i} = z_i \frac{e_i v_i}{l_i l_{0_i}^2} (l_i - l_{0_i}), \quad (2)$$

$$\bar{\lambda}_i = \frac{\|\bar{t}_i\|_2}{\bar{l}_i} = z_i \frac{e_i v_i}{\bar{l}_i l_{0_i}^2} (\bar{l}_i - l_{0_i}) \quad (3)$$

Lengths of the elements l_i, \bar{l}_i , at configurations p and $p + u$ respectively are computed as,

$$l_i = \|g_i\|_2, \quad \bar{l}_i = \|\bar{g}_i\|_2, \quad \bar{g} = M(p + u), \quad (4)$$

$z \in \mathbf{R}^{n_{el}}$ is a vector whose i^{th} entry is 1 or -1 if i^{th} element is a string or bar respectively end $e \in \mathbf{R}^{n_{el}}$ is a vector of Young's modulus of the elements. The constraint $\lambda_i \geq 0$ is equivalent to

$$-z_i(l_i - l_{0_i}) \leq 0. \quad (5)$$

2.2 Large Displacement Static Response of the Structure

Once the force f is applied on the structure that is properly supported and constrained from rigid body motion, its static response u is computed from the changed force balance equation in the configuration $p + u$,

$$C \hat{\lambda} M(p + u) + f + f_c = 0, \quad C_1 u = c_1. \quad (6)$$

$\bar{\lambda}$ is defined in (2) and $f_c \in \mathbf{R}^{3n_n}$ is unknown nodal constraint reaction force, $C_1 \in \mathbf{R}^{3n_n \times 3n_n}$ and $c_1 \in \mathbf{R}^{3n_n}$ are given to constrain nodal displacements as a result of the presence of the supports. Note that these two constraints can jointly be handled by deleting the rows of the force balance equation involving constraint forces.

Since string elements can not transmit compressive load their element force in the loaded configuration $p + u$ must not reverse its sign. In other words strings must remain stretched, which is written in the following form,

$$-z_i(\bar{l}_i - l_{0_i}) \leq 0, \quad i \in I_s, \quad (7)$$

where I_s is a set of indices of string elements.

2.3 Design Constraints

A natural constraint on rest length of all the elements requires that $l_{0_i} > 0$. Moreover, for them to be suitable

for manufacturing a more restrictive constraint has to be imposed requiring,

$$l_{0_i} \geq l_{0_{min_i}} > 0, \quad (8)$$

for a given vector $l_{0_{min}} \in \mathbf{R}^{n_{el}}$.

2.3.1. Strength Constraint All element stresses in both loaded and unloaded configuration must not exceed their yield stress value. Since all the elements are axially loaded this constraint takes the following form:

$$z_i e_i (l_i - l_{0_i}) - l_{0_i} \sigma_i \leq 0, \quad (9)$$

$$z_i e_i (\bar{l}_i - l_{0_i}) - l_{0_i} \sigma_i \leq 0. \quad (10)$$

Note that this constraint is directly derived from the Hooke's law that relates stress of an axially loaded linear elastic element to relative deformation of the element,

$$\sigma_i = z_i e_i \frac{(l_i - l_{0_i})}{l_{0_i}} \quad (11)$$

Since bar forces may reverse their direction in the loaded configuration they are additionally constrained by,

$$e_i (\bar{l}_i - l_{0_i}) - l_{0_i} \sigma_i \leq 0, \quad i \in I_b \quad (12)$$

where I_b is a set of bar indices.

2.3.2. Buckling Constraint To ensure that stability of the elements is preserved under the prestress forces and after an external load is applied, bar elements have to be constrained from buckling. The bar forces in case of a linear elastic material model must not exceed values given by Euler's formula. Expressing bar forces in terms of force coefficients and corresponding element lengths this requirement is written as:

$$\lambda_i l_i \leq \|t_{i_{critical}}\|_2, \quad \bar{\lambda}_i \bar{l}_i \leq \|\bar{t}_{i_{critical}}\|_2, \quad i \in I_b \quad (13)$$

$$\|t_{i_{critical}}\|_2 = \frac{\pi^2 e_i I_{i_{min}}}{l_{0_i}^2},$$

where $I_{i_{min}}$ is a minimal moment of inertia of the cross section of the element. Assuming that all bar elements have a round cross section $I_{i_{min}}$ is computed as,

$$I_{i_{min}} = \frac{v_i^2}{4\pi l_{0_i}^2}, \quad (14)$$

so that the buckling constraint finally becomes:

$$-l_{0_i}^2 (l_i - l_{0_i}) - \frac{\pi}{4} v_i \leq 0, \quad i \in I_b \quad (15)$$

$$-l_{0_i}^2 (\bar{l}_i - l_{0_i}) - \frac{\pi}{4} v_i \leq 0, \quad i \in I_b. \quad (16)$$

2.3.3. Shape Constraints To ensure that designed tensegrity structure can be supported at certain available locations and that the load acting at specified location can be attached to it, the unloaded tensegrity

structure has to satisfy shape constraints written in the following linear form,

$$C_2 p = c_2 \quad (17)$$

where $C_2 \in \mathbf{R}^{3n_n \times 3n_n}$, $c_2 \in \mathbf{R}^{3n_n}$ are given.

If a designed tensegrity structure is to be controlled by changing its element lengths, it is necessary to constrain their minimal length to make room for actuators to be attached. This constraint also guarantees that Jacobian of the constraints is well defined because, as it will be shown, it involves inverses of the lengths of the elements.

3. NONLINEAR PROGRAM FORMULATION

Now that the objective function and all the constraints are defined the optimization problem is written as:

Given C , M , I_s , I_b , z , e , f , σ , v_{total} , l_{0min} , l_{min}

$$\begin{aligned} & \min_{p, l_0, v, u} \frac{1}{2} f^T u \quad s.t. \\ & C \hat{\lambda} M p = 0 \\ & \lambda_i = l_i^{-1} l_{0i}^{-2} z_i e_i v_i (l_i - l_{0i}), \quad l_i = \|g_i\|_2, \quad g = M p, \\ & C_1 p = c_1, \\ & C \hat{\lambda} M (p + u) + f = 0, \\ & \bar{\lambda}_i = \bar{l}_i^{-1} \bar{l}_{0i}^{-2} z_i e_i v_i (\bar{l}_i - l_{0i}), \\ & \bar{l}_i = \|\bar{g}_i\|_2, \quad \bar{g} = M (p + u), \\ & C_2 u = c_2, \\ & -z_i (l_i - l_{0i}) \leq 0, \\ & l_{0i} - \bar{l}_i \leq 0, \quad i \in I_s, \\ & -l_{0i} + l_{0min_i} \leq 0, \\ & -l_i + l_{min_i} \leq 0, \\ & -\bar{l}_i + \bar{l}_{min_i} \leq 0, \\ & z_i e_i (l_i - l_{0i}) - l_{0i} \sigma_i \leq 0, \\ & z_i e_i (\bar{l}_i - l_{0i}) - l_{0i} \sigma_i \leq 0, \\ & e_i (\bar{l}_i - l_{0i}) - l_{0i} \sigma_i \leq 0, \quad i \in I_b, \\ & -l_{0i}^2 (l_i - l_{0i}) - \frac{\pi}{4} v_i \leq 0, \quad i \in I_b, \\ & -\bar{l}_{0i}^2 (\bar{l}_i - l_{0i}) - \frac{\pi}{4} v_i \leq 0, \quad i \in I_b, \\ & -v_i \leq 0, \\ & \|v\|_1 - v_{total} = 0. \end{aligned}$$

The design variables enter the constraints in a very similar way. It is easy to see for example, that they appear similarly in the minimal length and the strength constraints, up to a multiplication with e_i . This property simplifies analytical computation of the constraint Jacobian and suggests how to scale the problem to improve its condition since it otherwise might be ill-conditioned if realistic material data were used.

The solutions of the problem defined are obtained using SNOPT 6.1 (Gill *et al.*, 1999) software for large scale sparse nonlinear optimization that uses a sequential quadratic programming (SQP) method.

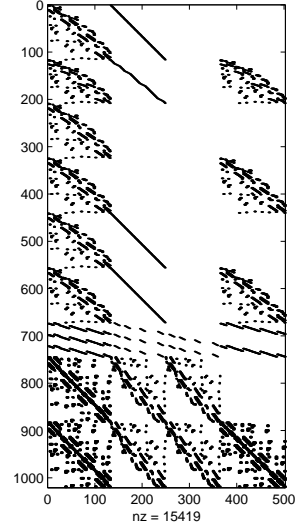


Fig. 1. Sparsity pattern of the the analytically computed constraint Jacobian for the given 3D example

4. NUMERICAL RESULTS

The basic 2D problem used to illustrate the optimal mass to stiffness ratio tensegrity design is given in Figure 2.

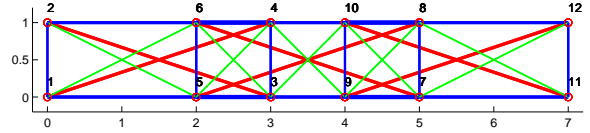


Fig. 2. Initial not optimized tensegrity beam design in unloaded state; unstressed element: light gray, pre-stressed bars: dark gray, pre-stressed tendons: black

This tensegrity beam is:

- built up from 3 planar tensegrity crosses,
- with an aspect ratio of 7,
- supported at the nodes 1 and 2, and
- is loaded by a unit vertical load at the top right node 12.

After applying the optimization procedure the following design illustrated in Figure 3 is obtained.

We note the following:

- the optimum tends to include class two elements (*i.e.* nodes where two bars are connected) because some nodes move close to each other.
- requirement for the strings to be uncompressed in the loaded configuration is sometimes a binding constraint yielding an optimal structure that has slack strings at loaded configuration.
- the same optimum is obtained consistently when starting from different initial conditions.

Enumerative types of design studies (de Jager and Skelton, 2001) tend to conclude that for optimal stiffness this structure has long bars, forming almost a

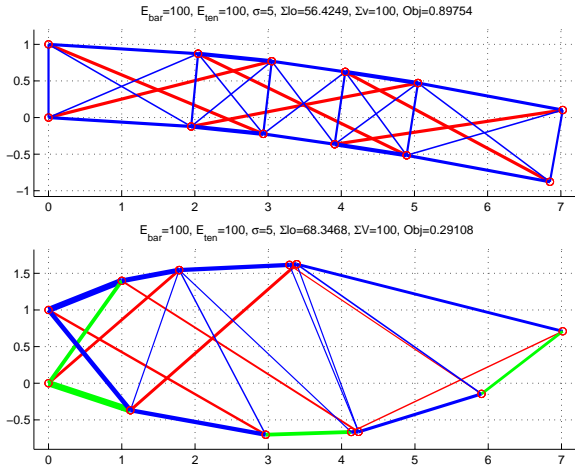


Fig. 3. Initial vs. optimized tensegrity beam design in loaded state, showing deformation under load; unstressed element: light gray, pre-stressed bars: dark gray, pre-stressed tendons: black

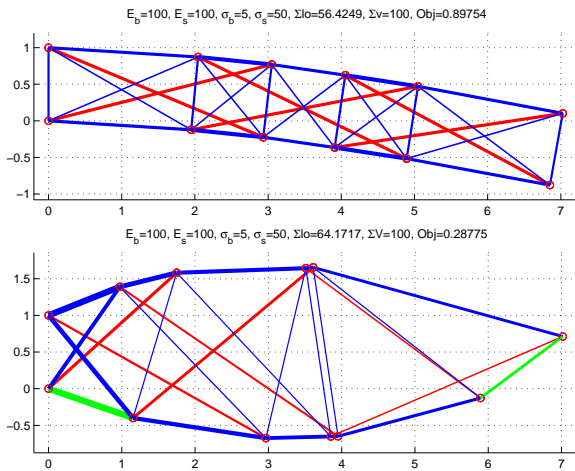


Fig. 4. Initial vs. optimal design when string material is stronger than bar material ; unstressed element: light gray, pre-stressed bars: dark gray, pre-stressed tendons: black

super tensegrity cross, while for optimal mass-to-stiffness ratio, the bars tend to be much shorter. This study did not incorporate significant changes in geometry, nor did it include constraints for failure of the structure. Here these shortcomings are readdressed.

From the example shown in Figure 4 where a different materials are used for bars and strings it appears that the results are sensitive to the different parameter choice as could be expected.

To make a comparison of this NLP approach, to an optimal mass-to-stiffness ratio truss design (Jarre *et al.*, 1998) that can be formulated as an LP-problem, a solution obtained using the latter formulation is given in Figure 5.

In Figure 6 a 3D design example is illustrated. This tensegrity structure is made of four 6-bar tensegrity units. It is supported at nodes 1, 6, 11 and loaded at the opposite end at the node 44 with a vertical unit force.

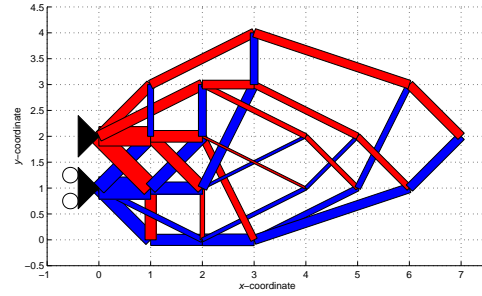


Fig. 5. Optimal topology starting from a dense grid; compressive load: dark gray, tensile load: black
It seems that the geometry of the optimal design tends to mimic the planar design by concentrating most of the elements in one plane.

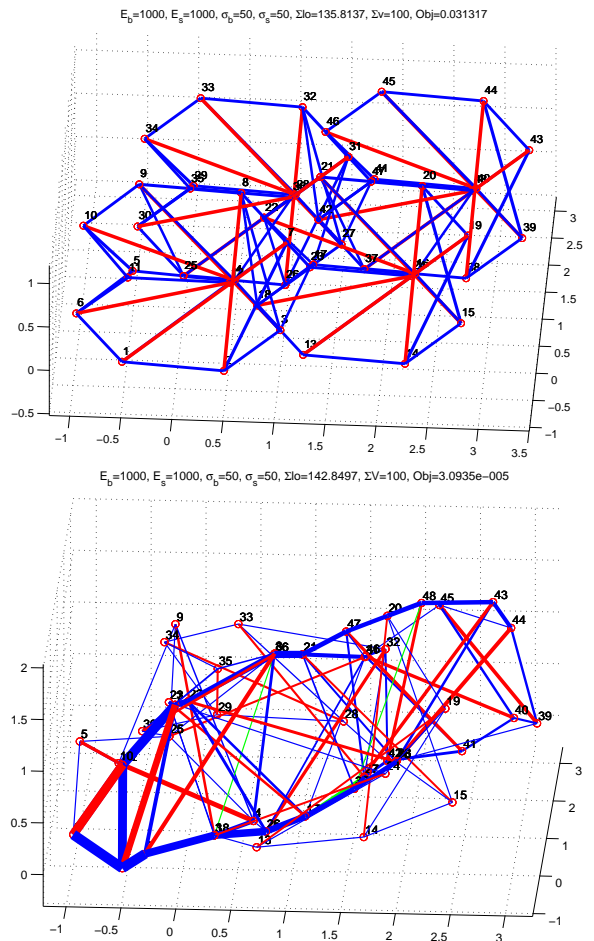


Fig. 6. Initial vs. optimal design of a 3D tensegrity beam system in loaded state; unstressed element: light gray, pre-stressed bars: dark gray, pre-stressed tendons: black

5. CONCLUSION AND FURTHER RESEARCH GOALS

The conclusions are as follows:

- tensegrity topological and geometrical optimization, cast in the form of a nonlinear program, is effectively solvable,

- if the problem is feasible the optimization approach is an appropriate design tool that guarantees monotonic stiffness improvement compared to an initial design,
- class two tensegrity topologies are advantageous,
- optimal topologies are highly asymmetric.

5.1 Further research

The following needs to be added to the problem formulation to better fulfill practical requirements of a designer of controlled tensegrity systems:

- include symmetry in the design objective by penalizing utilization of too many different elements to reduce manufacturing expenses
- optimize parameterized geometry which guarantees desired level of symmetry
- add buckling constraint for the structure as a whole.

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