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Equilibria and Stiffness of Planar Tensegrity Structures

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ABSTRACT

Tensegrity is a light weight deployable structure that is composed of compressive and tensile members. However, the compressive members, which are made of common metallic structural materials, are not mass-efficient. In this paper, we replace the compressive member with "self-similar" tensegrity structure that has same strength but less mass. We start with a non-tensegrity structure called $C4T1^i$ and show that it can be designed to have the same strength but less weight compared with the compressive member to be replaced. The stiffness-to-mass ratio of the $C4T1^i$ structure, however, is compromised. We then modify the $C4T1^i$ structure to a tensegrity structure called $C4T2^i$ by adding another set of string(s) with pre-stress. It can be shown that $C4T2^i$ is a better substitute of the compressive members in a structure because of less mass and same strength but higher stiffness-to-mass ratio.

Keywords: Tensegrity, Deployable Structure.

INTRODUCTION

Class 1 tensegrity is a strong and lightweight structure built by a set of axially loaded compressive members[1] —- bars, which do not touch[2] each other but are joined by some continuous network of pre-stressed tensile members —- strings[1]. The word *pre-stress* means the tensile members are under tension when the structure is in stable equilibrium state without external load[2, 3, 4, 5]. This can be expressed as

$$\mathbf{A}(\mathbf{q})\,\mathbf{t}(\mathbf{q}) = \mathbf{0}, \qquad \|\,\mathbf{t}(\mathbf{q})\,\| > \mathbf{0}, \tag{1}$$

where $\mathbf{A}$ is the connectivity matrix of tensile members, $\mathbf{t}$ is the tension vector of these members and $\mathbf{q}$ is the generalized coordinate. We call (1) the tensegrity condition. Both compressive and tensile members are complementary to each other and participate against external load. The presence of the strings enable us to adjust the tensions using actuators or controllable materials like shape memory alloy (SMA), and hence change the shape of the structure. One application of tensegrity is large deployable space structures that have a small stowed volume[6].

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[1]We assume all the compressive members are Euler columns in this paper
[2]Only end to end contact

If one reduces the length of a tensile member in the structure to zero, equation (1) can still be satisfied, treating the contact as node. To classify the tensegrity in such situations, Skelton introduces the definition **Class-k Tensegrity**[7] to enrich the variety of tensegrity. Generally, a class-k tensegrity structure is defined as, at maximum, k compressive members touch each other at a single node.

Given a tensegrity structure A, we would like to ask the question: Can we further improve[8, 9, 10, 11] the mass and stiffness properties? To answer this question, we first look at the reasons why tensegrity can be strong and lightweight. In building a tensegrity structure, tensile members are introduced and pre-stressed to resist external loads. Tensile members are usually more efficient in mass than compressive members. For this reason, tensegrity structures are lightweight compared with classical structures. Therefore, if the use of compressive members is minimized while the use of tensile members is maximized in the existing tensegrity structure, we may be able to improve the strength and reduce the weight of the structure. In this case, we need to design a tensegrity structure B such that it has at least same strength but less mass compared with the compressive members (to be replaced) in the existing tensegrity structure A.

After we replace the compressive members by another tensegrity structure B, the resultant structure contains new compressive members(from tensegrity B). This procedure results in a self-similar structure and we coin the name ***Self-Similar Tensegrity*** to describe this feature. With the self-similar idea, the structural member can be scaled down for easier manufacturing and construction. In control, less energy is required for actuators to do the structural deployment or stiffness control[12, 13, 14, 6, 11].

We first introduce a planar self-similar non-tensegrity structure called $C4T1^i$ that replaces a compressive member(a single bar). We will show how to design the $C4T1^i$ structure while the mass is reduced but the strength is preserved compared to the single bar. Load deflection curves and stiffness will be computed. We then modify the $C4T1^i$ structure by adding string(s) in suitable positions to obtain a self-similar tensegrity structure called $C4T2^i$. In particular, we will show that the $C4T2^i$ structure can be used to replace the compressive member, as we can reduce the mass and increase the stiffness-to-mass ratio. The remarkable feature of the design is that we start with a non-tensegrity structure $C4T1^i$ to obtain the design information of bars and strings and the design information can then be applied directly to the tensegrity structure $C4T2^i$.

The notation "$CxTy^i$" means a structure composed of x **C**ompressive members and y **T**ension members. The index i denotes the total number of self-similar stages/iterations. When $i = 0$, no self-similar iteration is taken place. For $i \geq 1$, new members will be introduced in every iteration and we use subscript, for example $j = 1, 2, \ldots, i$, to represent the members introduced in the j-th iteration. For materials, the notations ρ_i, ρ_{t_j}, E_i, E_{t_j} and σ_{t_j} represent the density of bars, string, Young's modulus of bars, string and yield stress of string respectively. For length quantities, we use l_i, and l_{t_j} to represent the length of bars and strings respectively. Except the length of the structure l_0, the extra subscript '0' of the length quantities like $l_{i,0}$ and $l_{t_j,0}$ represents the rest length of that members. The length-to-diameter ratio of bars and strings are denoted as ξ_i and ξ_{t_j} respectively. In particular, we will use capital notation to represent the length and force quantities of structural members at buckling or yield. For example, L_i and R_i represent the length and radius of a bar in i-th iteration when the buckling load, F_i, of the bar is applied. The length and buckling load of the whole structure will be represented by L_0 and F_0 respectively.

All the compressive and tensile members are assumed to be cylindrical and have constant stiffness under deformation. Hence, Hooke's law will be applied in the analysis.

In the numerical computation of the structure, we will assume both compressive members and tensile members are made of steel, which has the density $\rho = 7.862\ g/cm^3$, Young's modulus $E = 2.06 \times 10^{11}\ N/m^2$ and Yield strength $\sigma = 6.9 \times 10^8\ N/m^2$. Also, the angles (as defined later) α_j in every stage is assumed the same. i.e $\alpha_j = \alpha$ where α is some constant. The length of structures at buckling, L_0, will be normalized in the numerical calculations. i.e. $L_0 = 1$.

MOTIVATION

Considering an Euler column, we call it as the "original bar" such that it buckles at the compressive load F_0. For the original bar of radius R_0 and length L_0 with the buckling load F_0 applied, the mass of the bar $m_0 = m_{b_0}$ is

$$m_0 = m_{b_0} = \rho_0 \pi R_0^2 L_0 = \frac{\rho_0 \pi}{4\xi_0^2} L_0^3. \tag{2}$$

The buckling load of the bar F_0 is given by the Euler's buckling formula

$$F_0 = \frac{E_0 \pi^3 R_0^4}{4L_0^2} = \frac{E_0 \pi^3}{64\xi_0^4} L_0^2. \tag{3}$$

Alternatively, the buckling load F_0 can be written in terms of the mass m_0 as

$$F_0 = \frac{E_0 \pi m_0^2}{4\rho_0^2 L_0^4}. \tag{4}$$

Now, consider a simple stable class 1 tensegirty of 2 bars and 4 strings as shown in figure 1. We call this $C2T4$ and would like to design the structure such that it buckles as the same buckling load as the original bar while less mass is required. Hence $F_0/2$ is applied on each nodal point when the structure is buckle and the horizontal strings will be slack for mass minimization. Each bar will carry the compressive load of $F_1 = F_0/2\cos\theta$. With the similar buckling formula (4) applied to each bar, we have the total mass of 2 bars

$$m_b = \sqrt{\frac{E_0}{E_1}} \frac{\rho_1}{\rho_0} \frac{m_0}{\cos\theta}.$$

Therefore, if the material of bars are the same as the original bar, we cannot obtain a less mass structure

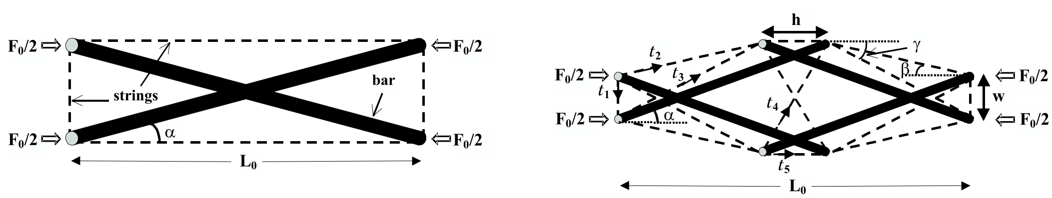

FIG. 1. Left: A $C2T4$ Structure, Right: A $C4T14$ Structure

while preserving the strength. But the ability of lateral load resistance drawn us the interest to re-design the structure. Consider a structure $C4T14$ made by the connection of 2 $C2T4$ units as shown in figure 1, equilibrium condition gives

$$\frac{F_0}{2} + t_2 \cos\beta + t_3 \cos\gamma = F_1 \cos\alpha. \tag{5}$$

With the same argument as before, strings 2 and 3 should be slack ($t_2 = t_3 = 0$) when the buckling load F_0 is applied on each side. Therefore, the compressive load of each bar is $F_1 = F_0/2\cos\theta$. With (4) and applying the buckling formula to bar, we obtain the total mass of 4 bars

$$m_b = \sqrt{8\frac{E_0}{E_1}\left(\frac{\rho_1}{\rho_0}\right)\cos\alpha} \frac{L_1}{2L_1\cos\alpha - h} m_0 \geq \sqrt{\frac{E_0}{E_1}\left(\frac{\rho_1}{\rho_0}\right)\frac{1}{2\cos^5\alpha}} m_0.$$

So, the minimal mass structure is obtain when the horizontal overlap $h = 0$ and is independent of the vertical overlap $w = 0$. If the same material is used to make the bars, we may be able to construct the $C4T14$ structure with less mass compared with the original bar. With the consideration of robustness to lateral load and self-similar construction (as discussed later), we would set $w = 0$ and hence a rhombus structure called $C4T1$ is obtained.

$C4T1^i$ STRUCTURE

With the idea of the $C4T1$ structure, we would like to design a class of self-similar structure called $C4T1^i$ to replace the original bar such that the structure buckles at the same load F_0 (buckling constraint) but yet requires less mass (minimal mass design for a fixed geometry). Due to the requirement of minimal mass design, the word "buckled" used for the new structure means all the compressive and tensile members will buckle and yield at precisely the load F_0.

$C4T1^1$

Setting both the vertical w and horizontal h overlap to zero, we obtain a new 4 bars/1 string structure called $C4T1^1$ that replaces the original bar as shown in Figure 2. We call this step (to replace a single bar) the first iteration of the self-similar procedure and hence put an index 1 to the structural name. The angle α_1 is the maximum angle between the bar and the axis parallel to applied force F_0 without buckling.

The total mass m_{b_1} of the 4 bars is

$$m_{b_1} = 4\rho_1\pi R_1^2 L_1 = \frac{\rho_1\pi}{\hat{\xi}_1^{\,2}} L_1^3, \tag{6}$$

and the buckling load of each bar F_1 is

$$F_1 = \frac{E_1\pi^3 R_1^4}{4L_1^2} = \frac{E_1\pi^3}{64\xi_1^4} L_1^2 = \frac{E_1\pi m_{b_1}^2}{64\rho_1^2 L_1^4}, \tag{7}$$

where, from the geometry of the structure,

$$L_1 = \frac{L_0}{2\cos\alpha_1}, \quad F_1 = \frac{F_0}{2\cos\alpha_1}. \tag{8}$$

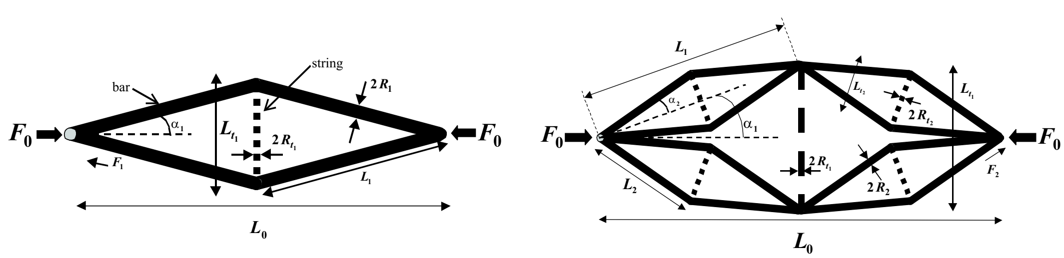

FIG. 2. Left: A $C4T1^1$ Structure, Right: A $C4T1^2$ Structure

Mass of Bars in a $C4T1^1$ Structure

Since the new structure is subjected to the buckling constraint, we can express the mass of 4 bars m_{b_1} to that of the original bar m_0 as

$$m_{b_1} = \frac{\rho_1}{\rho_0}\sqrt{\left(\frac{E_0}{E_1}\right)\left(\frac{1}{2\cos^5\alpha_1}\right)}m_0. \tag{9}$$

We can see that the mass of bars in the $C4T1^1$ structure is minimum when $\alpha_1 = 0°$ and increased as α_1 increases. With the same materials as the original bar, the condition for the mass of bar $m_{b_1} < m_{b_0} = m_0$ is

$$\alpha < \alpha_{max} = \cos^{-1}(0.5)^{\frac{1}{5}} = 29.477°.$$

So, in order to have a lightweight structure (string is assumed to be massless in this section) which is made of the same materials and buckles at the same load F_0 as the original bar, $\alpha \leq 29.477°$. Note that the mass reduction is maximum (about 29.3%) when $\alpha = 0°$. But in this case, the structure has no robustness to lateral loads. Therefore, there is a tradeoff between the mass reduction of the bars and robustness to lateral loads.

Mass and Tension of String in a $C4T1^1$ Structure

The mass m_{t_1}, length L_{t_1}, and yield tension T_1 of the string in the $C4T1^1$ structure are

$$m_{t_1} = \rho_{t_1}\pi R_{t_1}^2 L_{t_1} = \frac{\rho_{t_1}\pi}{4\xi_{t_1}^2}L_{t_1}^3, \tag{10}$$

$$L_{t_1} = L_0\tan\alpha_1, \tag{11}$$

$$T_1 = F_0\tan\alpha_1 = \sigma_{t_1}\pi R_{t_1}^2 = \frac{\sigma_{t_1}m_{t_1}}{\rho_{t_1}L_{t_1}} = \frac{\sigma_{t_1}\pi}{4\xi_{t_1}^2}L_{t_1}^2, \tag{12}$$

where we have related T_1 to the yield stress of string materials σ_{t_1} in (12).

We can also relate the mass of string and the original bar as

$$m_{t_1} = \left(\frac{E_0\pi^2\tan^2\alpha_1}{16\sigma_{t_1}\xi_0^2}\right)\left(\frac{\rho_{t_1}}{\rho_0}\right)m_0. \tag{13}$$

The quantity m_{t_1} increases with α_1, which is expected since larger cross section area and longer string length are required in order to resist the external load F_0.

Aspect Ratio of Structural Members in a $C4T1^1$ Structure

With the information of buckling or yield load of structural members, the length-to-diameter ratio of these members can be written in term of ξ_0. For bars, the length-to-diameter ratio of bars, ξ_1, is

$$\xi_1 = \frac{L_1}{2R_1} = \left(\frac{E_1}{E_0}\right)^{\frac{1}{4}}\left(\frac{1}{2\cos\alpha_1}\right)^{\frac{1}{4}}\xi_0, \tag{14}$$

which is proportional to ξ_0 and increases with $\alpha_1 < \alpha_{max}$. It is less than ξ_0 if the bars are made of the same material as the original bar. Hence, thicker bars should be used to build the $C4T1^1$ structure. For string, the length-to-diameter ratio, ξ_{t_1}, is

$$\xi_{t_1} = \frac{L_{t_1}}{2R_{t_1}} = \frac{4}{\pi}\sqrt{\frac{\sigma_{t_1}}{E_0}\tan\alpha_1}\xi_0^2, \tag{15}$$

which increases with α_1.

Total Mass of a $C4T1^1$ Structure

From (9) and (13), the total mass of the $C4T1^1$ structure, m_1, is

$$m_1 = \left[\left(\frac{E_0}{E_1}\right)^{\frac{1}{2}} \left(\frac{\rho_1}{\rho_0}\right) \left(\frac{1}{2\cos^5\alpha_1}\right)^{\frac{1}{2}} + \left(\frac{E_0\pi^2\tan^2\alpha_1}{16\sigma_{t_1}\xi_0^2}\right)\left(\frac{\rho_{t_1}}{\rho_0}\right)\right] m_0. \tag{16}$$

Using bars and string of the same materials as the original bar, the minimal mass occurs at $\alpha = 0°$, yielding $m_1 = m_0/\sqrt{2}$. Hence, the $C4T1^1$ structure can be used to replace the original bar which preserves the strength and decreases mass. Because of the finite string mass, the upper bound of α is always less than $\alpha_{max} = 29.477°$ and depends on ξ_0 for mass reduction .

$C4T1^i$

Since mass reduction is possible for the replacement of the original bar by the $C4T1^1$ structure while the buckling load is kept the same, we can further reduce the mass of the structure by taking another iteration over the compressive members of the $C4T1^1$ structure, A $C4T1^2$ structure is shown in Figure 2. A self-similar structure, called $C4T1^i$ structure, can be obtained as before. Figure 3 shows the $C4T1^i$ structures for $i = 3$ to 6 and $\alpha_j = 15°$ where $j = 1,2,\ldots,i$. In With the similar analysis as before, we can relate the total mass of the

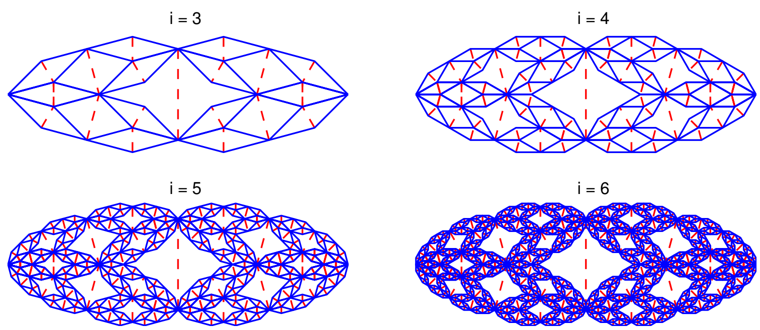

FIG. 3. Configurations of $C4T1^i$ Structures with $\alpha_j = 15°$, where $j = 1,2,\ldots,i$.

bars m_{b_i}, in the $C4T1^i$ structure to the mass of $C4T1^0$, m_0, as

$$m_{b_i} = \left(\frac{\rho_i}{\rho_0}\right)\sqrt{\left(\frac{E_0}{E_i}\right)\frac{1}{\prod_{j=1}^{i} 2\cos^5\alpha_j}}\, m_0. \tag{17}$$

For $\alpha_j < \alpha_{max}$, m_{b_i} is decreased with i and with α_j and we have the following theorem

Theorem 1 *Assume massless strings and valid buckling formula of bars over all iteration, then the mass of the $C4T1^i$ structure, with the buckling constraint design, approaches zero as i goes to infinity and the form of the resulting tensegrity fractals appears in figure 3.*

In practice, we can obtain arbitrary structural mass since we need to account for the mass of strings and the practical limit of self-similar iterations as discussed later.

Because of the buckling constraint design, the total mass of strings in the j-th iteration, m_{t_j}, can be related to m_0 as

$$m_{t_j} = \frac{E_0\pi^2\sin^2\alpha_j}{16\sigma_{t_j}\xi_0^2\prod_{r=1}^{j}\cos^2\alpha_r}\left(\frac{\rho_{t_j}}{\rho_0}\right) m_0. \tag{18}$$

As the iteration number i increases, the mass of strings increases. A decrease in the angle α_r ($r \leq j$) will decrease the string mass.

The length-to-diameter ratio of the bars, ξ_i, and strings in the j-th iteration, ξ_{t_j}, in the $C4T1^i$ structure are

$$\xi_i = \frac{L_i}{2R_i} = \left(\frac{E_i}{E_0}\right)^{\frac{1}{4}} \left(\frac{1}{\prod_{j=1}^{i} 2\cos\alpha_j}\right)^{\frac{1}{4}} \xi_0, \tag{19}$$

$$\xi_{t_j} = \frac{L_{t_j}}{2R_{t_j}} = \frac{4}{\pi}\sqrt{\left(\frac{\sigma_{t_j}}{E_0}\right)\left(\frac{2\sin\alpha_j}{\prod_{r=1}^{j} 2\cos\alpha_r}\right)}\xi_0^2. \tag{20}$$

The aspect ratio of both members are always increasing with increasing $\alpha_r < \alpha_{max}$, but decrease with an increasing of j. Hence, "fatter" bars and strings are required in the next iteration.

Practical Limit of Self-Similar Iteration

In the calculations of the $C4T1^i$ structure, it is assumed that all the compressive members are Euler columns. From (19), the length-to-diameter ratio of bars decreases with the iteration number and the Euler column assumption is no longer valid after a certain number of iterations. For a bar to be an Euler column, its buckling stress has to be less than or equal to its yield stress σ. i.e. $\frac{E\pi^2 R^2}{4L^2} < \sigma$, which gives the minimum length-to-diameter ratio

$$\xi_{min} = \frac{\pi}{4}\sqrt{\frac{E}{\sigma}}. \tag{21}$$

Using (19), we have the following theorem about the practical limit of self-similar iterations.

Theorem 2 *For an original Euler bar, with length-to-diameter ratio ξ_0, Young's Modulus E and yield stress σ, to be replaced by a $C4T1^i$ structure subject to the minimal mass design and buckling constraint, the iteration number is bounded by $\lfloor i_{limit} \rfloor$, where*

$$i_{limit} = 4\frac{\ln(\xi_0/\xi_{min})}{\ln(2\cos\alpha)}. \tag{22}$$

The ξ_{min} is given by (21). The materials of the bars in the $C4T1^i$ structure is assumed to be the same as the original bar and $\alpha_j = \alpha$ for $j = 1, 2, \ldots, i$.

Proof: Solve for i from the inequality $\xi_{min} \leq \xi_i$ using (19) and (21) with $E_0 = E_i$, $\alpha_j = \alpha$ for $j = 1, 2, \ldots, i$. $\square$

Total Mass of $C4T1^i$ Structure

The total mass m_i of the $C4T1^i$ structure, from (17) and (18), will be

$$m_i = \left[\left(\frac{E_0}{E_i}\right)^{\frac{1}{2}}\left(\frac{\rho_i}{\rho_0}\right)\left(\frac{1}{\prod_{j=1}^{i} 2\cos^5\alpha_j}\right)^{\frac{1}{2}} + \sum_{j=1}^{i}\frac{E_0\pi^2\sin^2\alpha_j}{16\sigma_{t_j}\xi_0^2\prod_{r=1}^{j}\cos^2\alpha_r}\left(\frac{\rho_{t_j}}{\rho_0}\right)\right] m_0. \tag{23}$$

If the materials of the bars and strings are the same as the original bar and $\alpha_j = \alpha$ for $j = 1, 2, \ldots, i$, then, the total mass can be simplified to

$$m_i = \left[\left(\frac{1}{2\cos^5\alpha}\right)^{\frac{i}{2}} + \frac{E_0\pi^2}{16\sigma_t\xi_0^2}\left(\frac{1}{\cos^{2i}\alpha} - 1\right)\right] m_0. \tag{24}$$

Figures 4 shows a plot of mass ratio m_i/m_0 versus the iteration number i for different α and $\xi_0 = 100$. As α increases, the mass contribution of strings to the structure is greater and leads to a heavier structure for fixed i. In particular, for $\alpha \geq 29.24°$, the mass reduction of bars cannot compensate for the increase in string mass with increasing i and hence, no mass reduction. For $\alpha < 29.24°$, the mass of the bars decreases as the iterations number increases but this is offset by the increase of string mass as seen in (17) and (18). This implies that the string mass will lower the upper bound value of α for mass reduction. Maximum mass

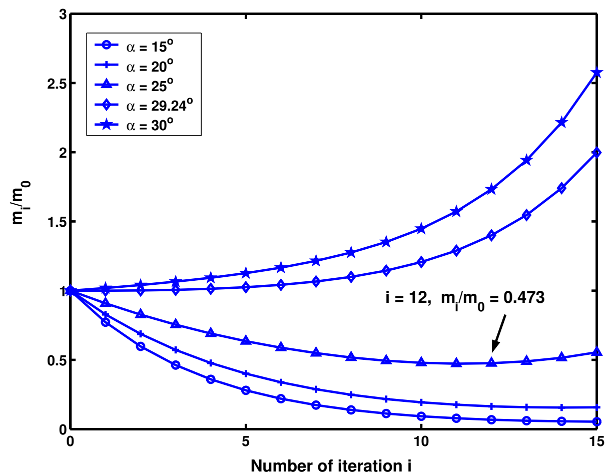

FIG. 4. Mass Ratio m_i/m_0 versus Iteration Number i for $\xi_0 = 100$

reduction can be achieved in some finite iteration number, we call it the *optimal* iteration, which depends on the angle α and ξ_0. For example, the optimal iteration number is 12 and the mass reduction is 52.7% for $\alpha = 25°$ and $\xi_0 = 100$. In fact, from (24), the mass reduction will be maximum when the iteration number i is given by the following theorem:

Theorem 3 *For a $C4T1^i$ structure subject to a buckling constraint, if the materials of bar and strings are the same in the $C4T1^i$ structure and $\alpha_j = \alpha$ for $j = 1, 2, \ldots, i$, minimum mass is achieved at a finite number of iterations and this number is given by either $\lceil i_{opt} \rceil$ or $\lfloor i_{opt} \rfloor$, where*

$$i_{opt} = \frac{2}{\ln\left(\frac{1}{2\cos\alpha}\right)} \ln\left(\frac{E_0 \pi^2}{4\sigma_t \xi_0^2} \frac{\ln\cos\alpha}{\ln\frac{1}{2\cos^5\alpha}}\right). \tag{25}$$

One must check the mass at both $\lceil i_{opt} \rceil$ and $\lfloor i_{opt} \rfloor$ to choose the smallest mass.

Proof: Take the derivative of m_i/m_0 in (24) with respect to i and set it equal to zero. Solving the resulting equation for i yields (25). □

Figure 5 shows a plot of the practical limit i_{limit} (dash curves) and the optimal i_{opt} (solid curves) iteration versus angle α given by equations (22) and (25). At about $\alpha < 24°$, we cannot obtain a minimal mass structure due to practical limit of Euler's assumption. On the other hand, a minimal mass structure is achievable for about $\alpha > 24°$, since the iteration number required is less than the practical limit of iteration. For these reasons, the shaded region is the practical region of the $C4T1^i$ structural design for $\xi_0 = 60$ for example. For these step curves, there exists a range of α which gives the same iteration number. We know from the previous calculation that the total mass of the $C4T1^i$ structure is decreased with the decrease of α, therefore, for a fixed iteration number, smallest angle should be chosen for smallest mass.

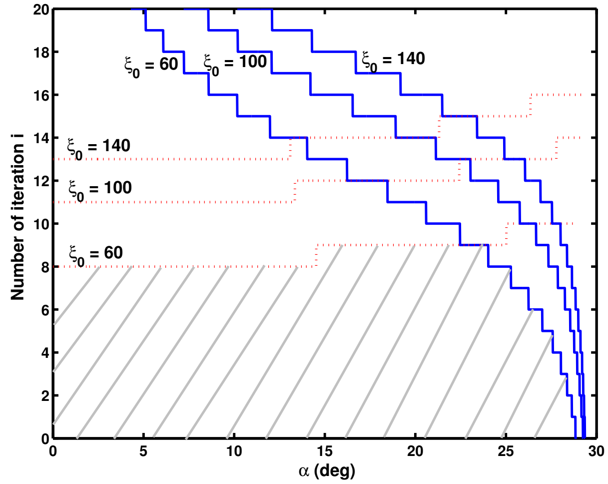

FIG. 5. Optimal (Solid Curves) and Practical Limit (Dash Curves) Iteration Number versus α

Load Deflection Curve of $C4T1^i$ Structure

In this paper, it is assumed that the stiffness of bars, k_{b_i}, and strings, k_{t_j}, where $1 \leq j \leq i$, are constant under deformation and measured at buckling for bars and yield for strings. i.e,

$$k_{b_i} = \frac{E_i A_i}{L_i} = \frac{E_i \pi R_i^2}{L_i}, \quad k_{t_j} = \frac{E_{t_j} A_{t_j}}{L_{t_j}} = \frac{E_{t_j} \pi R_{t_j}^2}{L_{t_j}}. \tag{26}$$

Hence,

$$f_i = k_{b_i} \left(l_{i,0} - l_i \right), \quad t_j = k_{t_j} \left(l_{t_j} - l_{t_j,0} \right). \tag{27}$$

The following theorem is needed to compute the load deflection curve,

Theorem 4 *For a $C4T1^i$ structure with the applied external force $0 \leq f_0 \leq F_0$, the force to length ratio of any member bar and string are equal to that of the structure. i.e.*

$$\frac{f_0}{l_0} = \frac{f_i}{l_i} = \frac{t_j}{l_{t_j}} \qquad j = 1, 2, \ldots, i, \tag{28}$$

where all the length, tension and compressive load quantities are measured under the external load f_0.

The proof is given in Appendix I. Note that $t_j = 0$ iff $f_0 = 0$, so the $C4T1^i$ structure cannot be pre-stressed and hence is not tensegrity.

From (27) and (28), the load deflection relation is

$$f_0 = k_{t_1} \left(1 - \frac{l_{t_1,0}}{l_{t_1}} \right) l_0. \tag{29}$$

A relationship between l_{t1} and l_0 is required to make the load deflection plot. Applying (27) and (28) in the geometry equation

$$l_0^2 = 4^i l_i^2 - \sum_{j=1}^{i} 4^{j-1} l_{t_j}^2, \tag{30}$$

we obtain

$$l_0^2 = 4^i \left(\frac{l_{i,0}}{1 + \frac{k_{t_1}}{k_{b_i}} \left(1 - \frac{l_{t_1,0}}{l_{t_1}} \right)} \right)^2 - \sum_{j=1}^{i} 4^{j-1} \left(\frac{l_{t_j,0}}{1 - \frac{k_{t_1}}{k_{t_j}} \left(1 - \frac{l_{t_1,0}}{l_{t_1}} \right)} \right)^2, \quad (31)$$

where k_{t_1}/k_{t_j}, k_{t_j}/k_{b_i}, $l_{i,0}$ and $l_{t_j,0}$ are given in Appendix III. Clearly, the inequality

$$l_{t_1,0} \leq l_{t_1} \leq L_{t_1}, \quad (32)$$

must be hold. For this range of values of l_{t_1}, one can find the corresponding values of l_0 and f_0 by (29) and (31), respectively. Hence, an indirect relation between f_0 and l_0 can be obtained.

Figure 6 shows the load deflection curves of the $C4T1^i$ structure with $\xi_0 = 100$ and $\alpha = 25°$ for different iteration number i. It is easier to deploy the structure with larger i. In the buckling design, the length of strings

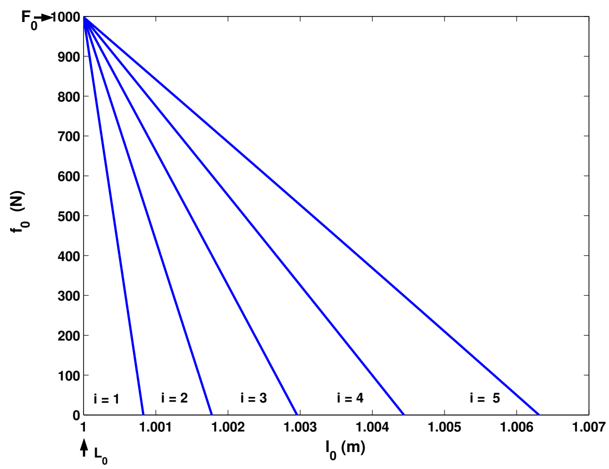

FIG. 6. Load Deflection Curves of $C4T1^i$ Structure with $\xi_0 = 100$ and $\alpha = 25°$

are only allowed to change in a very small range due to the small ratio σ_t/E_t in our choice of materials. Hence, the curves are quite linear and the structural stiffness is quite constant over the range of l_0.

Stiffness of the $C4T1^i$ Structure

Stiffness Definition

With the external force[3], the $C4T1^i$ structure changes its length (measured from 2 nodal points where external applied) in the same direction as the applied force. Therefore, the stiffness calculation is one-dimensional problem. For an external load f_0 applied to the structure of length l_0, the stiffness k of the structure is defined as

$$k = -\frac{df_0}{dl_0}, \quad (33)$$

where the negative sign means the length of the structure decreases as the applied load increases.

[3]Only axial force is considered since $C4T1^i$ structure is used to replace compressive member

The Stiffness Equation of a $C4T1^i$ Structure

The stiffness of $C4T1^i$, k_{C4T1^i}, (see Appendix II for proof) is given by

$$k_{C4T1^i} = k_{t_1}\left\{\left(\frac{l_{t_1,0}}{l_{t_1}} - 1\right) + \left[4^i \frac{k_{t_1}}{k_{b_i}}\frac{l_i}{l_{i,0}}\left(\frac{l_i}{l_0}\right)^2 + \sum_{j=1}^{i} 4^{j-1}\frac{k_{t_1}}{k_{t_j}}\frac{l_{t_j}}{l_{t_j,0}}\left(\frac{l_{t_j}}{l_0}\right)^2\right]^{-1}\right\}. \tag{34}$$

If $\alpha_j = \alpha$ and the materials of the bars and strings are the same as the original bar, the stiffness equation at buckling, K_{C4T1^i}, can be simplified to

$$K_{C4T1^i} = k_{t_1}\left\{-\frac{\sigma_t}{E_t} + \left[4^i \frac{k_{t_1}}{k_{b_i}}\frac{L_i}{l_{i,0}}\left(\frac{L_i}{L_0}\right)^2 + \left(1 - \frac{\sigma_t}{E_t}\right)^{-1}\left(\frac{1}{\cos^{2i}\alpha} - 1\right)\right]^{-1}\right\}, \tag{35}$$

where the physical quantities are given by Appendix III with $E_i = E_0$, $E_{t_j} = E_t$, $\sigma_{t_j} = \sigma_t$ and $\alpha_j = \alpha$.

The Rigid Bar Case

If the bars are infinitely rigid (large compared to the stiffness of strings), the ratio $k_{t_1}/k_{b_i} \to 0$. This is quite true if ξ_0 or α is large. We can also choose the materials such that the ratio σ_t/E_t is large. The stiffness equation (35) then becomes

$$K_{C4T1^i} = k_{t_1}\left(\frac{\cos^{2i}\alpha - \frac{\sigma_t}{E_t}}{1 - \cos^{2i}\alpha}\right). \tag{36}$$

The stiffness decreases with the increase in α and the iteration number, since longer strings (large α) or more strings (large i) makes the structure softer. Note that the ratio K_{C4T1^i}/k_{t_1} is independent of ξ_0, which is not true in the case of elastic bars, as discussed next. Since the stiffness reduces with each iteration i, it is of interest to know how many iterations may be taken before the stiffness violates a desired lower stiffness bound $\underline{K}$.

Theorem 5 *Given α and a desired lower stiffness bound $\underline{K}$ of a $C4T1^i$ structure subject to minimal mass design with buckling constraint, if the materials of bar and strings are the same as the original bar and $\alpha_j = \alpha$, the iteration number which achieves the stiffness requirement $\underline{K} \le K_{C4T1^i}$ with rigid bar assumption is bounded by $\lfloor i \rfloor$ where*

$$i \le \frac{1}{2\ln\cos\alpha}\ln\frac{\frac{\underline{K}}{k_{t_1}} + \frac{\sigma_t}{E_t}}{1 + \frac{\underline{K}}{k_{t_1}}}. \tag{37}$$

Proof: From (36), solving the inequality $\underline{K} \le K_{C4T1^i}$ for i gives (37). □

The Elastic Bar Case

If ξ_0 or σ_t/E_t is small, or the number of iteration is large in building the structure, the rigidity of bars in the $C4T1^i$ structure must be considered in the stiffness calculations, i.e. $k_{t_1}/k_{b_i} \neq 0$, and hence (35) should be used for stiffness calculations. Figure 7 shows a plot of stiffness at buckling, K_{C4T1^i}, versus α with $\xi_0 = 100$ for both elastic (solid curves) and rigid (dash curves) bars cases. The finite rigidity of bars results in lower stiffness compared with rigid bars case, especially in low value of α. The effect is smaller for large values of α because the ratio k_{t_1}/k_{b_i} is small in this range. Stiffness is less sensitive to geometry (choice of α) when bars are elastic.

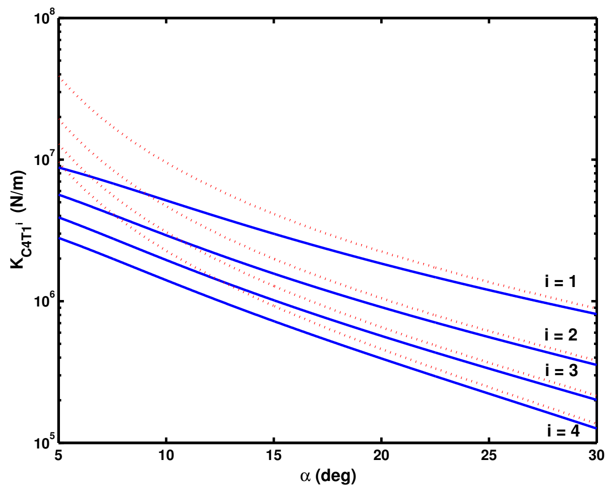

FIG. 7. Stiffness K_{C4T1^i} versus α for $\xi_0 = 100$, Elastic Bars Case (Solid Curves) and Rigid Bars Case (Dash Curves)

Stiffness Ratio and Stiffness-to-Mass Ratio

The stiffness of the $C4T1^0$ structure, at buckling, is

$$K_{C4T1^0} = \frac{E_0 \pi R_0^2}{L_0} = \frac{E_0 \pi L_0}{4\xi_0^2}. \tag{38}$$

With (35), the stiffness ratio K_{C4T1^i}/K_{C4T1^0} is always less than or equal to 1 and hence, no improvement is found in the stiffness of $C4T1^i$ structure. We can obtain the stiffness-to-mass ratio from (24) and (35). Calculations show that the best one can achieve, is the same ratio at $\alpha = 0°$ compared with the original bar. The ratio is getting smaller with the increase in α or iteration number. In conclusion, the $C4T1^i$ structure is less mass and is less stiff compared to the original bar.

$C4T2^i$ STRUCTURE

The $C4T1^i$ structure can be designed to have the same strength but less mass than the original bar. However, we still cannot improve the stiffness or stiffness-to-mass ratio of the structure. One way to solve this problem is to use pre-tension. In order to do this, consider a $C4T1^1$ structure with a horizontal string connected to both ends of the structure. This new structure is defined as $C4T2^1$. If we replace every compressive member by another smaller $C4T2^1$ structure and keep doing this in every iteration, we will obtain a $C4T2^i$ structure after i-th iteration. Compare with $C4T1^i$ structure, $C4T2^i$ structure can be pre-stressed. We shall call this new set of horizontal string(s) the h-string.

Buckling Design

One purpose of a $C4T2^i$ structure is to minimize the mass of the structure. Since the introduction of h-string(s) makes the structure heavier, h-strings are chosen such that all these strings will be slack if $f_0 = F_0$ is applied and will yield if $f_0 = 0$, so

$$t_{h_j} = \begin{cases} T_{h_j} = \sigma_{h_j}\pi R_{h_j}^2 & \text{for} \quad f_0 = 0 \quad j = 1,2,\ldots,i \\ 0 & \text{for} \quad f_0 = F_0. \end{cases} \tag{39}$$

Therefore, the pre-stress of an h-string is equal to its yield stress. Equation (39) implies that the rest lengths of h-strings are

$$l_{h_j,0} = L_{j-1} = \frac{L_0}{\prod_{r=1}^{j-1} 2\cos\alpha_r} \quad j = 1,2,\ldots,i. \tag{40}$$

Because of the design, all the structural members will be the same as that in the minimal mass design of the $C4T1^i$ structure with buckling constraints, except for the h-strings. Therefore, we only need to compute the geometry and intrinsic properties of the h-strings in order to satisfy the design.

Pre-Stress of h-Strings

The following analogy version of Theorem 3 is required for further calculations:

Theorem 6 *For $C4T2^i$ structure with the applied external force $0 \le f_0 \le F_0$, the force to length ratio of the structure is*

$$\frac{f_0}{l_0} = \frac{f_i}{l_i} - \sum_{s=1}^{i} \frac{t_{h_s}}{l_{h_s}} = \frac{t_j}{l_{t_j}} - \sum_{r=1}^{j} \frac{t_{h_r}}{l_{h_r}} \qquad j = 1,2,\ldots,i, \tag{41}$$

see Appendix IV for proof.

When $f_0 = 0$, there exists a set of $(f_i, t_{h_j}, t_{h_r}) \neq 0$ such that (41) is satisfied and hence the structure can be pre-stressed. There are many choices of pre-tension for the h-strings. In this paper, we choose the pre-tension in terms of buckling load F_0. For every h-string in j-th iteration, the pre-tension (yield strength) is chosen as

$$T_{h_j} = a_j F_{j-1} = a_j \frac{F_0}{\prod_{r=1}^{j-1} 2\cos\alpha_r}, \tag{42}$$

where $0 \le a_j < 1$ are some constants called pre-tension parameters. If $a_j = 0$ for all j, we will recover the $C4T1^i$ structure as h-strings are always slack and hence redundant.

Stiffness of h-Strings

Without external load, the $C4T2^i$ structure may under pre-tension. From (41),

$$\frac{f_{i,a}}{l_{i,a}} = \sum_{j=1}^{i} \frac{T_{h_j}}{L_{h_j}}, \quad \frac{t_{j,a}}{l_{t_j,a}} = \sum_{r=1}^{j} \frac{T_{h_r}}{L_{h_r}}, \tag{43}$$

where the extra subscript 'a' represents the physical quantities measured under pre-tension state, which gives (see Appendix V for proof)

$$l_{i,a} = \frac{l_{i,0}}{1 + \frac{k_{t_i}}{k_i}\left(1 - \frac{l_{t_i,0}}{l_{t_i,a}}\right)}, \tag{44}$$

$$l_{t_j,a} = \frac{l_{t_j,0}}{1 + \frac{a_{j+1}F_j}{k_{t_j}L_{h_{j+1}}} - \frac{k_{t_{j+1}}}{k_{t_j}}\left(1 - \frac{l_{t_{j+1},0}}{l_{t_{j+1},a}}\right)} \qquad j = 1,2,\ldots,i-1. \tag{45}$$

Including the geometrical constraint

$$L_{h_j}^2 = 4L_{h_{j+1}}^2 - l_{t_j,a}^2, \qquad L_{h_{i+1}} = l_{i,a} \qquad j = 1,2,\ldots,i, \tag{46}$$

we can express L_{h_j} for $j = 1,2,\ldots,i$ and $l_{t_r,a}$ for $r = 1,2,\ldots,i-1$ in terms of pre-tension parameters, a_j, and the length of strings in the i-th iteration at unloaded state $l_{t_i,a}$. Solving for $l_{t_i,a}$ from the equation

$$a_1 F_0 = k_{h_1}(L_{h_1} - L_0) = L_{h_1} k_{t_1}\left(1 - \frac{l_{t_1,0}}{l_{t_1,a}}\right), \tag{47}$$

which depends on a_j, by using Newton's method with (44)-(46), we can then obtain the length of h-strings at pre-stressed state L_{h_j} and the stiffness of the h-strings is given by

$$k_{h_j} = \frac{E_{h_j}\pi R_{h_j}^2}{L_{h_j}} = \frac{E_{h_j} a_j F_{j-1}}{\sigma_{h_j} L_{h_j}} = \frac{E_{h_j} a_j}{\sigma_{h_j} L_{h_j}} \frac{F_0}{\prod_{r=1}^{j-1} 2\cos\alpha_r} \quad j = 1,2,\ldots,i. \tag{48}$$

Mass of $C4T2^i$ Structure

After specifying a_j and obtaining the length of h-strings L_{h_j} numerically, the mass of all h-strings in the j-th iteration will be

$$m_{h_j} = 4^{j-1}\rho_{h_j}\pi R_{h_j}^2 L_{h_j} = 4^{j-1}\frac{E_0\pi^2 a_j}{16\sigma_{h_j}\xi_0^2\prod_{r=1}^{j-1} 2\cos\alpha_r}\left(\frac{\rho_{h_j}}{\rho_0}\right)\left(\frac{L_{h_j}}{L_0}\right) m_0, \tag{49}$$

which is linearly increasing with the pre-stress parameters a_j. Form the buckling design of $C4T2^i$ structure, except h-strings, all the bars and strings have the same design as that in the $C4T1^i$ structure. Therefore, with (23), the total mass of the $C4T2^i$ is

$$\begin{aligned} m_i \quad = \quad & \left[\left(\frac{E_0}{E_i}\right)^{\frac{1}{2}}\left(\frac{\rho_i}{\rho_0}\right)\left(\frac{1}{\prod_{j=1}^{i} 2\cos^5\alpha_j}\right)^{\frac{1}{2}} + \sum_{j=1}^{i}\frac{E_0\pi^2\sin^2\alpha_j}{16\sigma_{t_j}\xi_0^2\prod_{r=1}^{j}\cos^2\alpha_r}\left(\frac{\rho_{t_j}}{\rho_0}\right)\right. \\ & \left. + \sum_{j=1}^{i} 4^{j-1}\frac{E_0\pi^2 a_j}{16\sigma_{h_j}\xi_0^2\prod_{r=1}^{j-1} 2\cos\alpha_r}\left(\frac{\rho_{h_j}}{\rho_0}\right)\left(\frac{L_{h_j}}{L_0}\right)\right] m_0. \end{aligned} \tag{50}$$

Load Deflection Curves of $C4T2^i$ Structure

The load deflection curves of the $C4T2^i$ structure can be obtained, from (41) for $j = 1$, as

$$f_0 = k_{t_1}\left(1 - \frac{l_{t_1,0}}{l_{t_1}}\right) l_{h_1} - k_{h_1}(l_{h_1} - L_0). \tag{51}$$

The lengths of the central vertical and horizontal strings must be in the range

$$\begin{aligned} l_{t_1,0} \le l_{t_1,a} \le \quad & l_{t_1} \le \quad L_{t_1}, \\ L_0 \le \quad & l_{h_1} \le \quad L_{h_1}. \end{aligned} \tag{52}$$

Compared to the $C4T1^i$ structure in Figure 6, the load deflection curves for the $C4T2^i$ are also quite linear, but the flexibility of the structure is less than that of the $C4T1^i$ due to the presence of h-strings.

Stiffness of $C4T2^i$ Structure

From (51), the stiffness of the $C4T2^i$ structure is

$$k_{C4T2^i} = -\frac{df_0}{dl_0} = -\frac{df_0}{dl_{h_1}} = k_{t_1}\left(\frac{l_{t_1,0}}{l_{t_1}} - 1 - \frac{l_{h_1} l_{t_1,0}}{l_{t_1}^2}\frac{dl_{t_1}}{dl_{h_1}}\right) + k_{h_1}, \tag{53}$$

where the proof and the expression of dl_{t_1}/dl_{h_1} are provided in Appendix VI. Figure 8 shows the plot of stiffness-to-mass, at buckling, ratio versus α for $\xi_0 = 100$ and $a_j = 0.1$ where $j = 1,2,\ldots,i$. Even though the stiffness of the $C4T2^i$ structure is always less than the original bar, the stiffness-to-mass ratio can be better for some range of α and depending on the iteration i and a_j. Therefore, if we fix the mass of materials, the $C4T2^i$ structure would be a better choice of design of compressive members compared to a single bar.

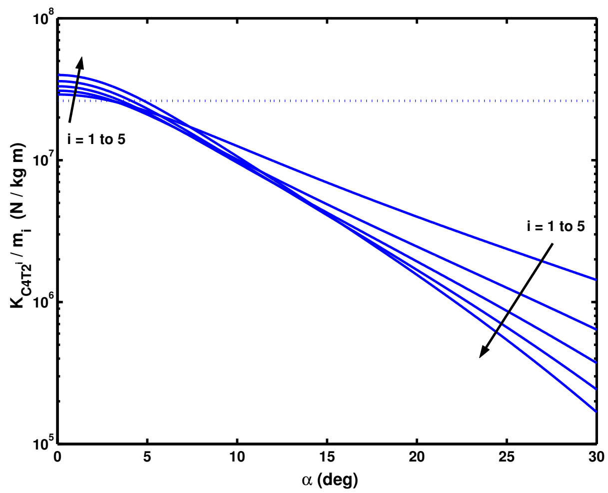

FIG. 8. Stiffness-to-Mass Ratio of a $C4T2^i$ Structure K_{C4T2^i}/m_i versus α for $\xi_0 = 100$ and $a_j = 0.1$ where $j = 1,2,\ldots,i$

CONCLUSIONS

For given structural materials like structural steel, long compressive members are usually not as mass efficient as tensile members. We attempt to design a tensegrity structure that replaces the compressive members with less mass while preserving strength. A self-similar structure called $C4T1^i$, which can distribute the external compressive load into both compression and tension of its structural members, is first proposed. Theoretical calculations show that less material is required to preserve strength. However, no improvement can be found in the stiffness or in the stiffness-to-mass ratio. Adding a set of horizontal strings in the $C4T1^i$ to make a new structure that can be pre-stressed, called $C4T2^i$, numerical calculations show that the $C4T2^i$ structure can be made stiffer by increasing the pre-tension. With a suitable pre-stressing condition, $C4T2^i$ not only inherits the advantages of the $C4T1^i$ structure, but also improves the stiffness-to-mass ratio compared to the original bar. This paper also gives an important motivation for tensegrity structures of class greater than class 1.

APPENDIX I. PROOF OF THEOREM 3

For a $C4T1^i$ structure under an external load $0 \le f_0 \le F_0$, the compressive load f_i of bars is

$$f_i = \frac{f_0}{\prod_{j=1}^{i} 2\cos\theta_j}. \tag{54}$$

where θ_j is the angle of the self-similar unit at j-stage due to the load f_0. If $f_0 = F_0$, $\theta_j = \alpha_j$. Similarly, the tension of strings in the i-th iteration is

$$t_j = \frac{2 f_0 \sin\theta_j}{\prod_{s=1}^{j} 2\cos\theta_s}, \qquad j = 1,2,3,...,i-1,i. \tag{55}$$

So, the load f_0 can be written in terms of any one of the compressive load of bar or tension of string

$$f_0 = f_i \prod_{s=1}^{i} 2\cos\theta_s = \frac{t_j}{2\sin\theta_j} \prod_{p=1}^{j} 2\cos\theta_p \quad \text{for } j = 1,2,3,...,i-1,i. \tag{56}$$

From the geometry of the structure,

$$\begin{aligned} \sin\theta_j &= \frac{l_{t_j}}{2l_j}, \\ \cos\theta_j &= \frac{l_{j-1}}{2l_j}. \end{aligned} \tag{57}$$

where

$$l_j = \frac{l_0}{\prod_{r=1}^{j} 2\cos\theta_r}. \tag{58}$$

Then, (56) can be simplified to

$$f_0 = f_i \frac{l_0}{l_i} = t_j \frac{l_0}{l_{t_j}}. \tag{59}$$

which is (28) as required.

APPENDIX II. DERIVATION OF STIFFNESS EQUATION FOR $C4T1^i$ STRUCTURE

From (27) and (28),

$$k_{b_i}\left(\frac{l_{i,0}}{l_i} - 1\right) = k_{t_j}\left(1 - \frac{l_{t_j,0}}{l_{t_j}}\right) = k_{t_1}\left(1 - \frac{l_{t_1,0}}{l_{t_1}}\right), \tag{60}$$

take the variation of all the length quantities gives

$$-k_{b_i}\frac{l_{i,0}}{l_i^2}\delta l_i = k_{t_j}\frac{l_{t_j,0}}{l_{t_j}^2}\delta l_{t_j} = k_{t_1}\frac{l_{t_1,0}}{l_{t_1}^2}\delta l_{t_1},$$

$$\begin{aligned} \delta l_i &= -\frac{k_{t_1} l_{t_1,0}}{k_{b_i} l_{i,0}} \frac{l_i^2}{l_{t_1}^2}\delta l_{t_1}, \\ \delta l_{t_j} &= \frac{k_{t_1} l_{t_1,0}}{k_{t_j} l_{t_j,0}} \frac{l_{t_j}^2}{l_{t_1}^2}\delta l_{t_1}. \end{aligned} \tag{61}$$

Taking the variation of the length of the structure form (30) and note that l_i is length of rods yields

$$\delta l_0 = 4^i \frac{l_i}{l_0}\delta l_i - \frac{1}{l_0}\sum_{j=1}^{i} 4^{j-1} l_{t_j}\delta l_{t_j}. \tag{62}$$

Combining (62) with (61) gives

$$\frac{dl_0}{dl_{t_1}} = -\frac{k_{t_1} l_{t_1,0}}{l_0 l_{t_1}^2}\left(4^i \frac{l_i^3}{k_{b_i} l_{i,0}} + \sum_{j=1}^{i} 4^{j-1}\frac{l_{t_j}^3}{k_{t_j} l_{t_j,0}}\right). \tag{63}$$

Computing the derivative of f_0 in (29) w.r.t. l_0 yields

$$\frac{df_0}{dl_0} = k_{t_1}\left(1 - \frac{l_{t_1,0}}{l_{t_1}}\right) + k_{t_1} l_0 \frac{l_{t_1,0}}{l_{t_1}^2}\frac{dl_{t_1}}{dl_0}, \tag{64}$$

and with (63), the stiffness of $C4T1^i$ is

$$\begin{aligned} k_{C4T1^i} = -\frac{df_0}{dl_0} &= k_{t_1}\left(\frac{l_{t_1,0}}{l_{t_1}} - 1\right) + l_0^2\left(4^i \frac{l_i^3}{k_{b_i} l_{i,0}} + \sum_{j=1}^{i} 4^{j-1}\frac{l_{t_j}^3}{k_{t_j} l_{t_j,0}}\right)^{-1} \\ &= k_{t_1}\left\{\left(\frac{l_{t_1,0}}{l_{t_1}} - 1\right) + \left[4^i \frac{k_{t_1}}{k_{b_i}}\frac{l_i}{l_{i,0}}\left(\frac{l_i}{l_0}\right)^2 + \sum_{j=1}^{i} 4^{j-1}\frac{k_{t_1}}{k_{t_j}}\frac{l_{t_j}}{l_{t_j,0}}\left(\frac{l_{t_j}}{l_0}\right)^2\right]^{-1}\right\}. \end{aligned} \tag{65}$$

APPENDIX III. SOME MATHEMATICAL RELATIONS IN BUCKLING DESIGN OF $C4T1^i$ STRUCTURE

The $C4T1^i$ system is designed to buckle at the same load as the original bar $C4T1^0$. The angles α_j where $j = 1, 2, ..., i-1, i$ are free variables to be specified to fix the geometry. Therefore, it is important to compute all the lengths and ratio quantities in terms of these angles.

Length of Structure and Strings

At buckling, he length of the structure and strings are

$$\begin{aligned} L_0 &= L_i \prod_{s=1}^{i} 2\cos\alpha_s, \\ L_{t_j} &= \frac{2L_0 \sin\alpha_j}{\prod_{r=1}^{j} 2\cos\alpha_r} = 2L_i \sin\alpha_j \prod_{s=j+1}^{i} 2\cos\alpha_s. \end{aligned} \tag{66}$$

Computing the Stiffness Ratio of Strings, $\frac{k_{t_s}}{k_{t_j}}$ where $s, j = 1, 2, 3, ..., i-1, i$

Consider the ratio

$$\frac{k_{t_{j+1}}}{k_{t_j}} = \frac{E_{t_{j+1}} A_{t_{j+1}}}{L_{t_{j+1}}}\frac{L_{t_j}}{E_{t_j} A_{t_j}} = \frac{E_{t_{j+1}}}{E_{t_j}}\left(\frac{\pi R_{t_{j+1}}^2}{\pi R_{t_j}^2}\right)\left(\frac{L_{t_j}}{L_{t_{j+1}}}\right). \tag{67}$$

With (66), the ratio can be simplified to

$$\frac{k_{t_{j+1}}}{k_{t_j}} = \frac{E_{t_{j+1}}}{E_{t_j}}\left(\frac{\sigma_{t_j}}{\sigma_{t_{j+1}}}\right).$$

Therefore, it is obvious that

$$\frac{k_{t_s}}{k_{t_j}} = \frac{E_{t_s}\sigma_{t_j}}{E_{t_j}\sigma_{t_s}}. \tag{68}$$

In particular, for the same materials of bars and strings,

$$\frac{k_{t_s}}{k_{t_j}} = 1. \tag{69}$$

this implies that the stiffness of all the members strings in a $C4T1^i$ structure are the same.

Computing the Stiffness Ratio of String to bar, $\frac{k_{t_j}}{k_{b_i}}$ where $j = 1,2,...,i-1,i$

From (68),

$$\frac{k_{t_j}}{k_{b_i}} = \frac{E_{t_j}\pi R_{t_j}^2}{L_{t_j}}\frac{L_i}{E_i\pi R_i^2} = \frac{E_{t_j}}{E_i}\left(\frac{\xi_i}{\xi_{t_j}}\right)^2\frac{L_{t_j}}{L_i}. \tag{70}$$

From (19), (20) and (66),

$$\frac{k_{t_j}}{k_{b_i}} = \frac{E_{t_j}\pi^2}{16\sigma_{t_j}\xi_0^2}\sqrt{\frac{E_0}{E_i}}\left(\prod_{s=1}^{i}2\cos\alpha_s\right)^{\frac{1}{2}}. \tag{71}$$

For $\alpha_j = \alpha$ and the same materials of bars and strings, (71) is reduced to

$$\frac{k_{t_j}}{k_{b_i}} = \frac{E_t\pi^2}{16\sigma_t\xi_0^2}(2\cos\alpha)^{\frac{i}{2}}. \tag{72}$$

Computing the Rest Length to Length Ratio of Strings, $\frac{l_{t_j,0}}{L_{t_j}}$

At buckling, the tension in the strings is given by

$$T_j = k_{t_j}\left(L_{t_j} - l_{t_j,0}\right) = \frac{E_{t_j}\pi R_{t_j}^2}{L_{t_j}}\left(L_{t_j} - l_{t_j,0}\right) = \frac{E_{t_j}T_j}{L_{t_j}\sigma_{t_j}}\left(L_{t_j} - l_{t_j,0}\right), \tag{73}$$

so

$$\frac{l_{t_j,0}}{L_{t_j}} = 1 - \frac{\sigma_{t_j}}{E_{t_j}}. \tag{74}$$

For the same materials of bars and strings,

$$\frac{l_{t_j,0}}{L_{t_j}} = 1 - \frac{\sigma_t}{E_t}. \tag{75}$$

Computing the Rest Length to Length Ratio of Bars $\frac{l_{i,0}}{L_i}$

From (28), at buckling,

$$\frac{F_i}{L_i} = \frac{T_1}{L_{t_1}}.$$

Hence, with (27)

$$k_{b_i}\left(\frac{l_{i,0}}{L_i} - 1\right) = k_{t_1}\left(1 - \frac{l_{t_1,0}}{L_{t_1}}\right). \tag{76}$$

Using (74), (76) is reduced to

$$\frac{l_{i,0}}{L_i} = 1 + \frac{k_{t_1}}{k_{b_i}} \frac{\sigma_{t_1}}{E_{t_1}} = 1 + \frac{\pi^2}{16\xi_0^2} \sqrt{\frac{E_0}{E_i}} \left(\prod_{s=1}^{i} 2\cos\alpha_s \right)^{\frac{1}{2}}. \tag{77}$$

For $\alpha_j = \alpha$ the same materials of bars and strings, with (72)

$$\frac{l_{i,0}}{L_i} = 1 + \frac{\pi^2}{16\xi_0^2} (2\cos\alpha)^{\frac{i}{2}}. \tag{78}$$

Computing the Stiffness of Central String, k_{t_1}

Recall that the stiffness of central string k_{t_1} is given by

$$k_{t_1} = \frac{E_{t_1} \pi R_{t_1}^2}{L_{t_1}}.$$

Using (3) and (66) the string stiffness becomes

$$k_{t_1} = \frac{E_{t_1}}{L_{t_1}} \frac{T_1}{\sigma_{t_1}} = \frac{E_{t_1}}{\sigma_{t_1}} \frac{F_0 \tan\alpha_1}{L_0 \tan\alpha_1} = \frac{E_0 E_{t_1} \pi^3 L_0}{64 \sigma_{t_1} \xi_0^4}. \tag{79}$$

For the same materials of bars and strings, (79) is reduced to

$$k_{t_1} = \frac{E_0^2 \pi^3 L_0}{64 \sigma_t \xi_0^4} = \frac{E_t^2 \pi^3 L_0}{64 \sigma_t \xi_0^4}. \tag{80}$$

APPENDIX IV. PROOF OF THEOREM 5

Consider the force equilibrium of the $C4T2^i$ structure, for bars, we have the following set of equations:

$$\begin{aligned} 2 f_i \cos\theta_i &= f_{i-1} + t_{h_i} \\ 2 f_{i-1} \cos\theta_{i-1} &= f_{i-2} + t_{h_{i-1}} \\ \vdots &= \vdots \\ 2 f_j \cos\theta_j &= f_{j-1} + t_{h_j} \\ \vdots &= \vdots \\ 2 f_1 \cos\theta_1 &= f_0 + t_{h_1}, \end{aligned} \tag{81}$$

where θ_j are the angles of the $C4T2$ unit in j-stage when the external load f_0 is applied.
So, the compressive load of every bar in the $C4T2^i$ structure is

$$f_i = \frac{f_0 + \sum_{j=1}^{i} t_{h_j} \prod_{s=1}^{j-1} 2\cos\theta_s}{\prod_{r=1}^{i} 2\cos\theta_r}. \tag{82}$$

For strings in j-th stage, equilibrium conditions yield

$$t_j = 2 f_j \sin\theta_j \qquad j = 1, 2, \ldots, i. \tag{83}$$

With (81), we have

$$t_j = 2\sin\theta_j \frac{f_0 + \sum_{r=1}^{j} t_{h_r} \prod_{s=1}^{r-1} 2\cos\theta_s}{\prod_{p=1}^{j} 2\cos\theta_p}. \tag{84}$$

From (82) and (84), we can express f_0 in terms of other quantities as

$$\begin{aligned} f_0 &= f_i \prod_{r=1}^{i} 2\cos\theta_r - \sum_{j=1}^{i} t_{h_j} \prod_{s=1}^{j-1} 2\cos\theta_s \\ &= t_j \frac{\prod_{p=1}^{j} 2\cos\theta_p}{2\sin\theta_j} - \sum_{r=1}^{j} t_{h_r} \prod_{s=1}^{r-1} 2\cos\theta_s. \end{aligned} \tag{85}$$

From the geometry,

$$\begin{aligned} \cos\theta_j &= \frac{l_{h_j}}{2 l_{h_{j+1}}}, \\ \sin\theta_j &= \frac{l_{t_j}}{2 l_{h_{j+1}}} \qquad j = 1, 2, \ldots, i-1, \end{aligned} \tag{86}$$

where $l_{h_{j+1}} = l_i$.
Substitute (86) into (85), we will have force to length ratio of the structure f_0/l_0 which is

$$\frac{f_0}{l_0} = \frac{f_i}{l_i} - \sum_{j=1}^{i} \frac{t_{h_j}}{l_{h_j}} = \frac{t_j}{l_{t_j}} - \sum_{r=1}^{j} \frac{t_{h_r}}{l_{h_r}} \qquad j = 1, 2, \ldots, i.$$

APPENDIX V. GEOMETRICAL RELATIONS IN $C4T2^i$ STRUCTURE

Under the assumptions of constant stiffness of bars and strings, (43) becomes

$$k_i \left(\frac{l_{i,0}}{l_{i,a}} - 1 \right) = \sum_{j=1}^{i} k_{h_j} \left(1 - \frac{l_{h_j,0}}{L_{h_j}} \right) = \sum_{j=1}^{i} k_{h_j} \left(1 - \frac{L_{j-1}}{L_{h_j}} \right), \tag{87}$$

$$k_{t_j} \left(1 - \frac{l_{t_j,0}}{l_{t_j,a}} \right) = \sum_{r=1}^{j} k_{h_r} \left(1 - \frac{l_{h_r,0}}{l_{h_r}} \right) = \sum_{r=1}^{j} k_{h_r} \left(1 - \frac{L_{r-1}}{L_{h_r}} \right). \tag{88}$$

From these equations, we have

$$\begin{aligned} k_i \left(\frac{l_{i,0}}{l_{i,a}} - 1 \right) &= k_{t_j} \left(1 - \frac{l_{t_j,0}}{l_{t_j,a}} \right), \\ l_{i,a} &= \frac{l_{i,0}}{1 + \frac{k_{t_i}}{k_i} \left(1 - \frac{l_{t_i,0}}{l_{t_i,a}} \right)}. \end{aligned} \tag{89}$$

Form (42) and (88), we have for r-th stage, where $r = 2, 3, \ldots, i$,

$$\begin{aligned} k_{h_r} \left(1 - \frac{l_{r-1}}{l_{h_r}} \right) &= k_{t_r} \left(1 - \frac{l_{t_r,0}}{l_{t_r,a}} \right) - k_{t_{r-1}} \left(1 - \frac{l_{t_{r-1},0}}{l_{t_{r-1},a}} \right), \\ \frac{a_r F_{r-1}}{L_{h_r}} &= k_{t_r} \left(1 - \frac{l_{t_r,0}}{l_{t_r,a}} \right) - k_{t_{r-1}} \left(1 - \frac{l_{t_{r-1},0}}{l_{t_{r-1},a}} \right), \\ l_{t_{r-1},a} &= \frac{l_{t_{r-1},0}}{1 + \frac{a_r F_{r-1}}{k_{t_{r-1}} L_{h_r}} - \frac{k_{t_r}}{k_{t_{r-1}}} \left(1 - \frac{l_{t_r,0}}{l_{t_r,a}} \right)}. \end{aligned} \tag{90}$$

APPENDIX VI. STIFFNESS EQUATION OF $C4T2^i$ STRUCTURE

From (53), we only need to compute $\frac{dl_{t_1}}{dl_{h_1}}$ in order to calculate the stiffness of $C4T2^i$ structure. With the assumptions of constant stiffness of bars and strings, (41) becomes

$$k_i\left(\frac{l_{i,0}}{l_i}-1\right)-\sum_{s=1}^{i}k_{h_s}\left(1-\frac{l_{h_s,0}}{l_{h_s}}\right)=k_{t_j}\left(1-\frac{l_{t_j,0}}{l_{t_j}}\right)-\sum_{r=1}^{j}k_{h_r}\left(1-\frac{l_{h_r,0}}{l_{h_r}}\right).$$

Taking the variation of all the length quantities yields

$$-\frac{k_i l_{i,0}}{l_i^2}\delta l_i-\sum_{s=1}^{i}\frac{k_{h_s}l_{h_s,0}}{l_{h_s}^2}\delta l_{h_s}=\frac{k_{t_j}l_{t_j,0}}{l_{t_j}^2}\delta l_{t_j}-\sum_{r=1}^{j}\frac{k_{h_r}l_{h_r,0}}{l_{h_r}^2}\delta l_{h_r}. \tag{91}$$

If we define

$$\begin{aligned} P_j &= \frac{k_{h_j}l_{h_j,0}}{l_{h_j}^2} \qquad j=1,2,\ldots,i,\\ Q_i &= \frac{k_i l_{i,0}}{l_i^2},\\ S_j &= \frac{k_{t_j}l_{t_j,0}}{l_{t_j}^2} \qquad j=1,2,\ldots,i, \end{aligned}$$

and (91) becomes

$$-Q_i\delta l_i-\sum_{s=1}^{i}P_s\delta l_{h_s}=S_j\delta l_{t_j}-\sum_{r=1}^{j}P_r\delta l_{h_r}. \tag{92}$$

For $j=i$ in (92), we get the differential relationship between l_i and l_{t_i} as

$$-Q_i\delta l_i=S_i\delta l_{t_i}. \tag{93}$$

For $j=i-1$ in (92), we have

$$-Q_i\delta l_i-P_i\delta l_{h_i}=S_{i-1}\delta l_{t_{i-1}}, \tag{94}$$

but from the geometry

$$l_{h_i}^2=4l_i^2-l_{t_i}^2,$$

and take the variation of length quantities gives

$$\begin{aligned} l_{h_i}\delta l_{h_i} &= 4l_i\delta l_i-l_{t_i}\delta l_{t_i},\\ \delta l_{h_i} &= \frac{4l_i}{l_{h_i}}\delta l_i-\frac{l_{t_i}}{l_{h_i}}\delta l_{t_i}, \end{aligned}$$

and with (93),

$$\delta l_i=\left[\frac{4l_i}{l_{h_i}}+\frac{l_{t_i}Q_i}{l_{h_i}S_i}\right]^{-1}\delta l_{h_i}. \tag{95}$$

So, (94) becomes

$$\begin{aligned} -\left\{\left[\frac{4l_i}{l_{h_i}}+\frac{l_{t_i}Q_i}{l_{h_i}S_i}\right]^{-1}Q_i+P_i\right\}\delta l_{h_i} &= S_{i-1}\delta l_{t_{i-1}},\\ -Q_{i-1}\delta l_{h_i} &= S_{i-1}\delta l_{t_{i-1}}, \end{aligned} \tag{96}$$

where we defined

$$Q_{i-1} = \left[\frac{4l_i}{l_{h_i}} + \frac{l_{t_i}Q_i}{l_{h_i}S_i}\right]^{-1} Q_i + P_i.$$

Generalize the above steps for $j = 1, 2, \ldots, i-2$ and we will obtain

$$\begin{aligned} -Q_j \delta l_{h_{i+1}} &= S_j \delta l_{t_j}, \\ Q_j &= \left(4\frac{l_{h_{j+2}}}{l_{h_{j+1}}} + \frac{l_{t_{j+1}}Q_{j+1}}{l_{h_{j+1}}S_{j+1}}\right)^{-1} Q_{j+1} + P_{j+1}. \end{aligned}$$

In summary, we have

$$\begin{aligned} \frac{dl_{t_1}}{dl_{h_1}} &= -\left(4\frac{l_{h_2}S_1}{l_{h_1}Q_1} + \frac{l_{t_1}}{l_{h_1}}\right)^{-1}, \\ Q_j &= \left(4\frac{l_{h_{j+2}}}{l_{h_{j+1}}} + \frac{l_{t_{j+1}}Q_{j+1}}{l_{h_{j+1}}S_{j+1}}\right)^{-1} Q_{j+1} + P_{j+1} \qquad j = 1, 2, \ldots, i-1, \\ Q_i &= \frac{k_i l_{i,0}}{l_i^2}, \\ S_j &= \frac{k_{t_j} l_{t_j,0}}{l_{t_j}^2} \qquad j = 1, 2, \ldots, i, \\ P_j &= \frac{k_{h_j} l_{h_j,0}}{l_{h_j}^2} \qquad j = 1, 2, \ldots, i, \\ l_{h_{i+1}} &= l_i. \end{aligned} \tag{97}$$

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