State Estimation with Finite Signal-to-Noise Models

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***Abstract*— This paper concerns the use of the finite signal-to-noise (FSN) model to design a state estimator. First, the sufficient conditions for the existence of the state estimator are provided. These conditions are expressed in terms of linear matrix inequalities (LMIs) and the parametrization of all the solutions is provided. Then an LMI based estimator design is addressed and the algorithm to solve the filtering problem is also given. The performance of the linear filter is examined by means of the numerical examples.**

I. Introduction

The traditional noise model used in estimation and control theory is white noise whose intensity is independent of the variance of the signal it is corrupting. However, in many engineering applications such as electromechanical systems, this traditional noise model has serious deficiencies. A new noise model, the Finite Signal-to-Noise (FSN) model, was proposed in [8], [11], [12] where the intensity of the noise corrupting the signal depends affinely on the variance of that signal.

FSN noise models are more practical than normal white noise models, since they allow the variances of the noises to be affinely related to the variances of the signals they corrupt. Such noises are found in digital signal processing with both fixed and floating point arithmetic. Such models are found in analog sensors and actuators which produce more noise when the power supplies in these devices must provide more power (for an increased dynamic range of the signals in the estimation or control problem).

One important benefit of the FSN model in a linear control problem is that it keeps the control finite at the maximal accuracy [7]. This is in contrast to LQG theory, where the maximal accuracy occurs at infinite control. Therefore, the FSN model has a significant effect on the robustness of the controller [5].

Recent studies also show the use of the FSN model for economic system design [3], [4]. Many engineering problems involve economic considerations, especially in mechanical and biochemical engineering. Assuming that the component cost is proportional to its signal-to-noise ratio, it is reasonable to integrate the instrumentation and control design, to obtain a low cost system for given performance requirements.

There has been great effort in recent years to provide a control theory for the FSN model. See [5], [6], [8] for a discussion of control problems with FSN noise models . Since this model reflects more realistic properties in engineering, as well as neuroscience [1], a complete theory which includes control and estimation should be developed.

This paper focuses on the study of estimation problem for the FSN model. Reference [12] demonstrates that the estimation problem is nonconvex. We shall show that a mild additional constraint for scaling will make the problem convex. The basic problem solved is to find a state estimator that bounds the estimation error below a specified error covariance.

The paper is organized as follows. In section 2, the estimation problem with the FSN model is formulated. In section 3, the sufficient conditions for the existence of the state estimator are given. Section 4 derives a linear estimator subject to a performance requirement. In section 5, a numerical example is presented, and the comparison between the FSN filter design and the Kalman filter is discussed.

The notation used in this paper is fairly standard. The transpose of a real matrix A is denoted by A^T; for symmetric matrix, the standard notation > 0 (≥ 0) is used to denote positive definite matrix (positive semi-definite matrix), and the notation < 0 (≤ 0) is used to denote negative definite matrix (negative semi-definite matrix); $\varepsilon(\cdot)$ denotes the expectation.

II. System Model and Problem Formulation

Consider the following linear system with state space representation

$$\dot{x} = Ax + D\omega, \tag{1}$$

$$y = Cx + v, \tag{2}$$

$$z = C_z x, \tag{3}$$

where $x \in R^{n_x}$ is state variable, $y \in R^{n_y}$ is the measurement output; $\omega \in R^{n_w}$ and $v \in R^{n_y}$ are zero-mean FSN white noises with intensities W, V respectively; and A, C and D are constant matrices which have proper dimensions.

Here we consider that the noise source is modeled according to the FSN assumption, where the intensity of the noise corrupting a signal is proportional to the intensity of that signal. That is, assuming

$$\varepsilon_\infty\{\omega(t)\} = 0, \;\; \varepsilon_\infty\{\omega(t)\omega(\tau)^T\} = W\delta(t-\tau), \tag{4}$$

$$\varepsilon_\infty\{v(t)\} = 0, \;\; \varepsilon_\infty\{v(t)v(\tau)^T\} = V\delta(t-\tau), \tag{5}$$

where $\varepsilon_\infty = lim_{t\rightarrow\infty}\varepsilon(\cdot)$, $\varepsilon(\cdot)$ is the expectation operator. Suppose the vector $\omega_a \in R^{n_w}$ describes the signal that is corrupted by the noise ω, and ω_a is linearly related to state variable x

$$\omega_a = Mx. \tag{6}$$

Define the state covariance matrix

$$X = \varepsilon_\infty\{x(t)x(t)^T\}, \tag{7}$$

therefore

$$\begin{aligned} W &= W_0 + \Sigma_\omega M X M^T \Sigma_\omega, && (8)\\ V &= V_0 + \Sigma_v C X C^T \Sigma_v, && (9) \end{aligned}$$

where W_0, V_0 are given positive definite constant matrices, and

$$\begin{aligned} \Sigma_\omega &= diag\{\sigma_{\omega_1}, \sigma_{\omega_2}, \ldots, \sigma_{\omega_{n_\omega}}\}, && (10)\\ \Sigma_v &= diag\{\sigma_{v_1}, \sigma_{v_2}, \ldots, \sigma_{v_{n_y}}\}, && (11) \end{aligned}$$

where $\sigma_{\omega_i}, \sigma_{v_i}$ are Noise-to-Signal Ratio (NSR) of the i^{th} channel respectively.

For this system, the objective is to design a linear filter with the state space representation

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + F(y - C\hat{x}), && (12)\\ \hat{z} &= C_z\hat{x}, && (13) \end{aligned}$$

where $\hat{x}$ is the estimate of the state x, F is the filter gain to be determined such that $(A - FC)$ is asymptotically stable, and the estimation error has covariance less than a specified matrix. The estimation error is $\tilde{x} = x - \hat{x}$, and the estimation error system is given by

$$\begin{aligned} \dot{\tilde{x}} &= (A - FC)\tilde{x} + D\omega - Fv, && (14)\\ \tilde{z} &= C_z\tilde{x}, && (15) \end{aligned}$$

where $\tilde{z}$ denotes the estimation error of particular interests. The key idea of this filtering problem is to find the estimate $\hat{x}$ of x such that the performance criterion $\varepsilon_\infty\{\tilde{z}\tilde{z}^T\} < \Omega$ is satisfied.

In this paper, the following two problems are analyzed. First, we will explore the existence condition of the state estimator. We will be able to provide the sufficient conditions for the existence of the state estimator based on Linear Matrix Inequalities (LMIs). Second, we will determine if there exists a filter gain F such that $\varepsilon_\infty\{\tilde{z}\tilde{z}^T\} < \Omega$ is satisfied for the given Ω.

III. Existence Condition

In the following we will consider the augmented estimation error dynamics

$$\dot{\mathbf{x}} = \mathcal{A}\mathbf{x} + \mathcal{D}\mathbf{w}, \tag{16}$$

where

$$\mathbf{x} = \begin{pmatrix} \tilde{x} \\ \hat{x} \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} \omega \\ v \end{pmatrix}, \tag{17}$$

$$\begin{aligned} \mathcal{A} &= \begin{pmatrix} A - FC & 0 \\ FC & A \end{pmatrix} = \mathcal{A}_0 + \mathcal{B}_0 F \mathcal{C}_0, && (18)\\ \mathcal{D} &= \begin{pmatrix} D & -F \\ 0 & F \end{pmatrix} = \mathcal{D}_0 + \mathcal{B}_0 F E_0, && (19) \end{aligned}$$

$$\mathcal{A}_0 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \qquad \mathcal{B}_0 = \begin{pmatrix} -I \\ I \end{pmatrix}, \tag{20}$$

$$\mathcal{C}_0 = \begin{pmatrix} C & 0 \end{pmatrix}, \qquad \mathcal{D}_0 = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}, \tag{21}$$

$$E_0 = \begin{pmatrix} 0 & I \end{pmatrix}. \tag{22}$$

<u>Definition 1</u> (**Mean Square Stable**) The error system (16) with FSN noise inputs is mean square stable if the state covariance matrix associated with this error system exists and is positive definite.

We start by defining the upper bound of the state covariance matrix of system (16) as

$$\mathcal{X} \geq \varepsilon_\infty\{\mathbf{x}(t)\mathbf{x}(t)^T\}, \tag{23}$$

if it exists, it satisfies the following inequality:

$$0 > \mathcal{X}\mathcal{A}^T + \mathcal{A}\mathcal{X} + \mathcal{D}\begin{pmatrix} W & 0 \\ 0 & V \end{pmatrix}\mathcal{D}^T. \tag{24}$$

Substitution of (8), (9), (18), (19) into the above inequality, yields

$$\begin{aligned} 0 \;>\; & \mathcal{X}(\mathcal{A}_0 + \mathcal{B}_0 F \mathcal{C}_0)^T + (\mathcal{A}_0 + \mathcal{B}_0 F \mathcal{C}_0)\mathcal{X} \\ & + \mathcal{N}_1 \mathcal{X} \mathcal{N}_1^T + (\mathcal{B}_0 F G_0)\mathcal{X}(\mathcal{B}_0 F G_0)^T \\ & + (\mathcal{D}_0 + \mathcal{B}_0 F E_0)\mathcal{W}_0(\mathcal{D}_0 + \mathcal{B}_0 F E_0)^T, \end{aligned} \tag{25}$$

where

$$\begin{aligned} \mathcal{N}_1 &= \begin{pmatrix} D\Sigma_\omega M & D\Sigma_\omega M \\ 0 & 0 \end{pmatrix}, && (26)\\ \mathcal{W}_0 &= \begin{pmatrix} W_0 & 0 \\ 0 & V_0 \end{pmatrix}, && (27)\\ G_0 &= \begin{pmatrix} -\Sigma_v C & -\Sigma_v C \end{pmatrix}. && (28) \end{aligned}$$

Lemma 1: (assume $\mathcal{W}_0 = I$) The inequality of (25) can be written as

$$\Gamma F \Lambda + (\Gamma F \Lambda)^T + \Theta < 0, \tag{29}$$

where

$$\Theta = \begin{pmatrix} \mathcal{A}_0\mathcal{X} + \mathcal{X}\mathcal{A}_0^T + \mathcal{N}_1\mathcal{X}\mathcal{N}_1^T & 0 & \mathcal{D}_0 \\ 0 & -\mathcal{X} & 0 \\ \mathcal{D}_0^T & 0 & -I \end{pmatrix}, \tag{30}$$

$$\Gamma = \begin{pmatrix} \mathcal{B}_0 \\ 0 \\ 0 \end{pmatrix}, \tag{31}$$

$$\Lambda = \begin{pmatrix} \mathcal{C}_0\mathcal{X} & G_0\mathcal{X} & E_0 \end{pmatrix}. \tag{32}$$

Proof: By using the Schur complement formula, the inequality (25) can be written as

$$\begin{pmatrix} \mathcal{X}\mathcal{A}^T + \mathcal{A}\mathcal{X} + \mathcal{N}_1\mathcal{X}\mathcal{N}_1^T & (\mathcal{B}_0 F G_0)\mathcal{X} & \mathcal{D}_0 + \mathcal{B}_0 F E_0 \\ \mathcal{X}(\mathcal{B}_0 F G_0)^T & -\mathcal{X} & 0 \\ (\mathcal{D}_0 + \mathcal{B}_0 F E_0)^T & 0 & -I \end{pmatrix} < 0,$$

where $\mathcal{A}$ is defined in (18). Breaking the above matrix into two matrices and substituting (30) into the above inequality, yields

$$\Theta + \begin{pmatrix} (\mathcal{B}_0 F \mathcal{C}_0)\mathcal{X} + \mathcal{X}(\mathcal{B}_0 F \mathcal{C}_0)^T & (\mathcal{B}_0 F G_0)\mathcal{X} & \mathcal{B}_0 F E_0 \\ \mathcal{X}(\mathcal{B}_0 F G_0)^T & 0 & 0 \\ (\mathcal{B}_0 F E_0)^T & 0 & 0 \end{pmatrix} < 0.$$

Rewrite the above inequality as the following

$$\Theta + \begin{pmatrix} \mathcal{B}_0 \\ 0 \\ 0 \end{pmatrix} F \begin{pmatrix} \mathcal{C}_0\mathcal{X} & G_0\mathcal{X} & E_0 \end{pmatrix}$$

$$+ \left(\begin{pmatrix} \mathcal{B}_0 \\ 0 \\ 0 \end{pmatrix} F \begin{pmatrix} \mathcal{C}_0\mathcal{X} & G_0\mathcal{X} & E_0 \end{pmatrix} \right)^T < 0,$$

By the definition of Γ and Λ given in the lemma, it yields

$$\Gamma F \Lambda + (\Gamma F \Lambda)^T + \Theta < 0.$$

■

In order to find the existence conditions of the state estimator and the parametrization of all the solutions, the following lemma from [9] can be applied.

Lemma 2 (Projection Lemma): Let Γ, Λ, Θ be given. There exists a matrix F satisfying $\Gamma F \Lambda + (\Gamma F \Lambda)^T + \Theta < 0$ if and only if the following two conditions hold

$$\Gamma^{\perp}\Theta\Gamma^{\perp^T} < 0, \tag{33}$$

$$\Lambda^{T^{\perp}}\Theta\Lambda^{T^{\perp^T}} < 0. \tag{34}$$

Supposed the above statements hold, then all the solution F are given by

$$F = -R^{-1}\Gamma^T\Phi\Lambda^T(\Lambda\Phi\Lambda^T)^{-1} + S^{1/2}L(\Lambda\Phi\Lambda^T)^{-1/2}, \tag{35}$$

where

$$S = R^{-1} - R^{-1}\Gamma^T[\Phi - \Phi\Lambda^T(\Lambda\Phi\Lambda^T)^{-1}\Lambda\Phi]\Gamma R^{-1}. \tag{36}$$

L is an arbitrary matrix such that $\|L\| < 1$ and R is an arbitrary positive definite matrix such that

$$\Phi = (\Gamma R^{-1}\Gamma^T - \Theta)^{-1} > 0. \tag{37}$$

Lemma 3: The condition (33) and (34) are equivalent to the following statement: there exist symmetric positive definite matrices $\mathcal{X}, P \in \mathbf{R}^{2n_x \times 2n_x}$ that satisfy

$$\mathcal{X}P = I, \tag{38}$$

$$\mathcal{B}_0^{\perp}(\mathcal{A}_0\mathcal{X} + \mathcal{X}\mathcal{A}_0^T + \mathcal{N}_1\mathcal{X}\mathcal{N}_1^T + \mathcal{D}_0\mathcal{D}_0^T)\mathcal{B}_0^{\perp^T} < 0, \tag{39}$$

$$\begin{pmatrix} \mathcal{C}_0^T \\ G_0^T \\ E_0^T \end{pmatrix}^{\perp} \begin{pmatrix} P\mathcal{A}_0 + \mathcal{A}_0^T P + P\mathcal{N}_1\mathcal{X}\mathcal{N}_1^T P & 0 & P\mathcal{D}_0 \\ 0 & -P & 0 \\ \mathcal{D}_0^T P & 0 & -I \end{pmatrix} \cdot \begin{pmatrix} \mathcal{C}_0^T \\ G_0^T \\ E_0^T \end{pmatrix}^{\perp^T} < 0. \tag{40}$$

Proof: The result follows from Lemma 1 and Projection Lemma where we note that

$$\Gamma^{\perp} = \begin{pmatrix} \mathcal{B}_0 \\ 0 \\ 0 \end{pmatrix}^{\perp} = \begin{pmatrix} \mathcal{B}_0^{\perp} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Substituting the above equation and (30) into (33), yields

$$\begin{pmatrix} \mathcal{B}_0^{\perp}(\mathcal{A}_0\mathcal{X} + \mathcal{X}\mathcal{A}_0^T + \mathcal{N}_1\mathcal{X}\mathcal{N}_1^T)\mathcal{B}_0^{\perp^T} & 0 & \mathcal{B}_0^{\perp}\mathcal{D}_0 \\ 0 & -\mathcal{X} & 0 \\ \mathcal{D}_0^T\mathcal{B}_0^{\perp^T} & 0 & -I \end{pmatrix} < 0.$$

A Schur complement of this matrix is

$$\begin{pmatrix} \mathcal{B}_0^{\perp}(\mathcal{A}_0\mathcal{X} + \mathcal{X}\mathcal{A}_0^T + \mathcal{N}_1\mathcal{X}\mathcal{N}_1^T)\mathcal{B}_0^{\perp^T} & 0 \\ 0 & -\mathcal{X} \end{pmatrix}$$

$$+ \begin{pmatrix} \mathcal{B}_0^{\perp}\mathcal{D}_0 \\ 0 \end{pmatrix} \begin{pmatrix} \mathcal{D}_0^T\mathcal{B}_0^{\perp^T} & 0 \end{pmatrix} < 0,$$

therefore

$$\begin{pmatrix} \mathcal{B}_0^{\perp}(\mathcal{A}_0\mathcal{X} + \mathcal{X}\mathcal{A}_0^T + \mathcal{N}_1\mathcal{X}\mathcal{N}_1^T + \mathcal{D}_0\mathcal{D}_0^T)\mathcal{B}_0^{\perp^T} & 0 \\ 0 & -\mathcal{X} \end{pmatrix} < 0,$$

which is equivalent to

$$\mathcal{B}_0^{\perp}(\mathcal{A}_0\mathcal{X} + \mathcal{X}\mathcal{A}_0^T + \mathcal{N}_1\mathcal{X}\mathcal{N}_1^T + \mathcal{D}_0\mathcal{D}_0^T)\mathcal{B}_0^{\perp^T} < 0, \ \mathcal{X} > 0.$$

Furthermore, since

$$\Lambda^{T^{\perp}} = \begin{pmatrix} \mathcal{X}\mathcal{C}_0^T \\ \mathcal{X}G_0^T \\ E_0^T \end{pmatrix}^{\perp} = \begin{pmatrix} \mathcal{C}_0^T \\ G_0^T \\ E_0^T \end{pmatrix}^{\perp} \begin{pmatrix} \mathcal{X}^{-1} & 0 & 0 \\ 0 & \mathcal{X}^{-1} & 0 \\ 0 & 0 & I \end{pmatrix},$$

defining $\mathcal{X}^{-1} = P$, and substituting (30) and the above equation into (34), yields

$$\begin{pmatrix} \mathcal{C}_0^T \\ G_0^T \\ E_0^T \end{pmatrix}^{\perp} \begin{pmatrix} P\mathcal{A}_0 + \mathcal{A}_0^T P + P\mathcal{N}_1\mathcal{X}\mathcal{N}_1^T P & 0 & P\mathcal{D}_0 \\ 0 & -P & 0 \\ \mathcal{D}_0^T P & 0 & -I \end{pmatrix} \cdot \begin{pmatrix} \mathcal{C}_0^T \\ G_0^T \\ E_0^T \end{pmatrix}^{\perp^T} < 0.$$

■

The above theorem provides the existence condition for the state estimator, and the characterization given in Lemma 3 is necessary and sufficient. However, we introduce a nonconvex constraint $\mathcal{X}P = I$, which makes our problem more difficult to solve. The next theorem shows how to write these conditions into convex constraints by using Finsler's Lemma from [9].

Lemma 4 (Finsler's Lemma): Let $x \in \mathbf{R}^n$, $\mathcal{Q} \in \mathbf{S}^n$ and $\mathcal{B} \in \mathbf{R}^{n\times m}$. Let $\mathcal{B}^{\perp}$ be any matrix such that $\mathcal{B}^{\perp}\mathcal{B} = 0$. The following statements are equivalent:

i) $x^T\mathcal{Q}x < 0,\ \forall \mathcal{B}^T x = 0,\ x \neq 0,$

ii) $\mathcal{B}^{\perp}\mathcal{Q}\mathcal{B}^{\perp^T} < 0,$

iii) $\exists \mu \in \mathbf{R} : \mathcal{Q} - \mu\mathcal{B}\mathcal{B}^T < 0,$

iv) $\exists \mathcal{Y} \in \mathbf{R}^{m\times n} : \mathcal{Q} + \mathcal{B}\mathcal{Y} + \mathcal{Y}^T\mathcal{B}^T < 0.$

Finsler's Lemma is a specialized version of the Projection Lemma, and it can be applied to obtain LMI formulations in control and estimation theory.

Theorem 1: There exists a state estimator gain F to solve (24) if there exist a symmetric matrix $P \in \mathbf{R}^{2n_x\times 2n_x}$ and $\mu_1 < 0, \mu_2 < 0 \in \mathbf{R}$ that satisfies

$$P > 0, \tag{41}$$

$$\begin{pmatrix} P\mathcal{A}_0 + \mathcal{A}_0^T P & P\mathcal{N}_1 & P\mathcal{D}_0 & P\mathcal{B}_0 \\ \mathcal{N}_1^T P & -P & 0 & 0 \\ \mathcal{D}_0^T P & 0 & -I & 0 \\ \mathcal{B}_0^T P & 0 & 0 & \mu_1 I \end{pmatrix} < 0, \tag{42}$$

$$\begin{pmatrix} P\mathcal{A}_0 + \mathcal{A}_0^T P & 0 & P\mathcal{D}_0 & \mathcal{C}_0^T & P\mathcal{N}_1 \\ 0 & -P & 0 & G_0^T & 0 \\ \mathcal{D}_0^T P & 0 & -I & E_0^T & 0 \\ \mathcal{C}_0 & G_0 & E_0 & \mu_2 I & 0 \\ \mathcal{N}_1^T P & 0 & 0 & 0 & -P \end{pmatrix} < 0. \tag{43}$$

Proof: The result follows from Lemma 3 and Finsler's Lemma. If the inequality (39) holds, it is equivalent to the following: there exists a $\mu_1 \in \mathbf{R}$ such that

$$\mathcal{A}_0\mathcal{X} + \mathcal{X}\mathcal{A}_0^T + \mathcal{N}_1\mathcal{X}\mathcal{N}_1^T + \mathcal{D}_0\mathcal{D}_0^T - \mu_1\mathcal{B}_0\mathcal{B}_0^T < 0.$$

Apply the congruence transformation

$$\mathcal{X}^{-1}(\mathcal{A}_0\mathcal{X} + \mathcal{X}\mathcal{A}_0^T + \mathcal{N}_1\mathcal{X}\mathcal{N}_1^T + \mathcal{D}_0\mathcal{D}_0^T - \mu_1\mathcal{B}_0\mathcal{B}_0^T)\mathcal{X}^{-1} < 0,$$

with $P := \mathcal{X}^{-1} > 0$, $\mu_1 < 0$ and the Schur complement, it provides the LMI

$$\begin{pmatrix} P\mathcal{A}_0 + \mathcal{A}_0^T P & P\mathcal{N}_1 & P\mathcal{D}_0 & P\mathcal{B}_0 \\ \mathcal{N}_1^T P & -P & 0 & 0 \\ \mathcal{D}_0^T P & 0 & -I & 0 \\ \mathcal{B}_0^T P & 0 & 0 & \mu_1 I \end{pmatrix} < 0. \tag{44}$$

If the inequality (40) holds, it is equivalent to the existence of a $\mu_2 \in \mathbf{R}$ such that

$$\begin{pmatrix} P\mathcal{A}_0 + \mathcal{A}_0^T P + P\mathcal{N}_1\mathcal{X}\mathcal{N}_1^T P & 0 & P\mathcal{D}_0 \\ 0 & -P & 0 \\ \mathcal{D}_0^T P & 0 & -I \end{pmatrix}$$

$$-\mu_2 \begin{pmatrix} \mathcal{C}_0^T \\ G_0^T \\ E_0^T \end{pmatrix} \begin{pmatrix} \mathcal{C}_0 & G_0 & E_0 \end{pmatrix} < 0.$$

Applying Schur complements twice with $\mu_2 < 0$, it obtains the following LMI

$$\begin{pmatrix} P\mathcal{A}_0 + \mathcal{A}_0^T P & 0 & P\mathcal{D}_0 & \mathcal{C}_0^T & P\mathcal{N}_1 \\ 0 & -P & 0 & G_0^T & 0 \\ \mathcal{D}_0^T P & 0 & -I & E_0^T & 0 \\ \mathcal{C}_0 & G_0 & E_0 & \mu_2 I & 0 \\ \mathcal{N}_1^T P & 0 & 0 & 0 & -P \end{pmatrix} < 0.$$

■

IV. FSN Filter Design

The key idea of the filtering problem to be addressed here is to determine a linear filter F such that the performance criterion, $\varepsilon_\infty\{\tilde{z}\tilde{z}^T\} < \Omega$, is satisfied. We know that $\varepsilon_\infty\{\tilde{z}\tilde{z}^T\}$ can be computed from

$$\varepsilon_\infty\{\tilde{z}\tilde{z}^T\} = C_z\,\varepsilon_\infty\{\tilde{x}\tilde{x}^T\}\,C_z^T = \bar{C}_z\mathcal{X}\bar{C}_z^{\,T}, \tag{45}$$

where the state covariance matrix $\mathcal{X}$ is defined in (23) and

$$\bar{C}_z = C_z \begin{bmatrix} I & 0 \end{bmatrix}. \tag{46}$$

The algorithm to solve the filtering problem can be derived from the following theorem.

Theorem 2: There exists a filter F such that $\varepsilon_\infty\{\tilde{z}\tilde{z}^T\} < \Omega$ if there exists a positive definite symmetric matrix $P \in \mathbf{R}^{2n_x\times 2n_x}$ and $\mu_1 < 0, \mu_2 < 0 \in \mathbf{R}$ that satisfy (42), (43) and

$$\begin{pmatrix} \Omega & \bar{C}_z \\ \bar{C}_z^{\,T} & P \end{pmatrix} > 0. \tag{47}$$

All the solutions F are given by

$$F = -R^{-1}\Gamma^T\Phi\Lambda^T(\Lambda\Phi\Lambda^T)^{-1} + S^{1/2}L(\Lambda\Phi\Lambda^T)^{-1/2}, \tag{48}$$

where

$$S = R^{-1} - R^{-1}\Gamma^T[\Phi - \Phi\Lambda^T(\Lambda\Phi\Lambda^T)^{-1}\Lambda\Phi]\Gamma R^{-1}. \tag{49}$$

L is an arbitrary matrix such that $\|L\| < 1$ and R is an arbitrary positive definite matrix such that

$$\Phi = (\Gamma R^{-1}\Gamma^T - \Theta)^{-1} > 0, \tag{50}$$

and

$$\Theta = \begin{pmatrix} \mathcal{A}_0 P^{-1} + P^{-1}\mathcal{A}_0^T + \mathcal{N}_1 P^{-1}\mathcal{N}_1^T & 0 & \mathcal{D}_0 \\ 0 & -P^{-1} & 0 \\ \mathcal{D}_0^T & 0 & -I \end{pmatrix},$$

$$\Gamma = \begin{pmatrix} \mathcal{B}_0 \\ 0 \\ 0 \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} \mathcal{C}_0 P^{-1} & G_0 P^{-1} & E_0 \end{pmatrix}.$$

Proof: The inequality (47) can be manipulated by

$$\varepsilon_\infty\{\tilde{z}\tilde{z}^T\} = \bar{C}_z\mathcal{X}\bar{C}_z^{\,T} < \Omega,$$

then we can use Schur complement to convert it into a LMI

$$\begin{pmatrix} \Omega & \bar{C}_z \\ \bar{C}_z^{\ T} & \mathcal{X}^{-1} \end{pmatrix} = \begin{pmatrix} \Omega & \bar{C}_z \\ \bar{C}_z^{\ T} & P \end{pmatrix} > 0.$$

And the proof for solving F follows a similar approach in [9]. ■

We observe that the optimization approach proposed in this theorem is a convex programming problem stated as LMIs, which can be solved by efficient methods.

V. Numerical Example

In order to determine the applicability of the method, an example to solve for the filter design is presented next. We will consider a simple mechanical system which consists of a mass, a spring and a damper. The plant noise and measurement noise are modelled as FSN white noise.

$$\begin{cases} \dot{x} & = & \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \omega, \\ y & = & \begin{pmatrix} 3 & 3 \end{pmatrix} x + v, \\ z & = & \begin{pmatrix} 1 & 1 \end{pmatrix} x. \end{cases}$$

For simplicity, we assume that $\Sigma_\omega = \sigma_\omega I$, $\Sigma_v = \sigma_v I$. The Noise-to-Signal Ratio (NSR) is $\sigma_\omega = 0.1$, $\sigma_v = 0.1$ respectively, and

$$M = \begin{pmatrix} 1 & 0.5 \end{pmatrix}.$$

The performance criterion for the filter design is $\varepsilon_\infty\{\tilde{z}\tilde{z}^T\} < \Omega$ where $\Omega = 4$.

The filter that results from our method is

$$\dot{\hat{x}} = \begin{pmatrix} -0.00895 & 0.99105 \\ -0.99494 & -0.99494 \end{pmatrix} \hat{x} + \begin{pmatrix} 0.002985 \\ -0.001685 \end{pmatrix} y.$$

The simulation result shows that the output covariance of the estimation error is

$$\varepsilon_\infty\{\tilde{z}\tilde{z}^T\} = 2.9585,$$

which satisfies the design requirement, since $2.9585 < 4$.

The performance of the FSN filter introduced in this paper is illustrated in Fig. 1, where the error of each state variable is plotted. When compared to Fig. 1, Fig. 2 demonstrates the inferiority of the Kalman filter, which ignores the FSN structure of the noise by setting $W = W_0$ and $V = V_0$ in (8) and (9). Note that the peak values of the state error using the standard Kalman filter (from Fig. 2) are approximately 38 and 23 respectively, as compared to peak errors of approximately 7 and 13 respectively for the FSN estimator.

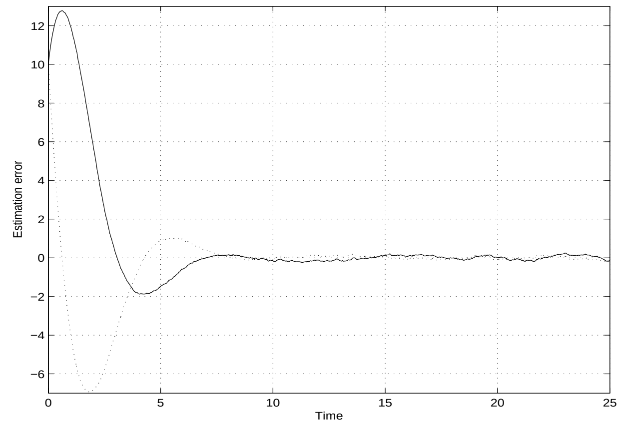

Fig. 1. Estimation error of FSN filter

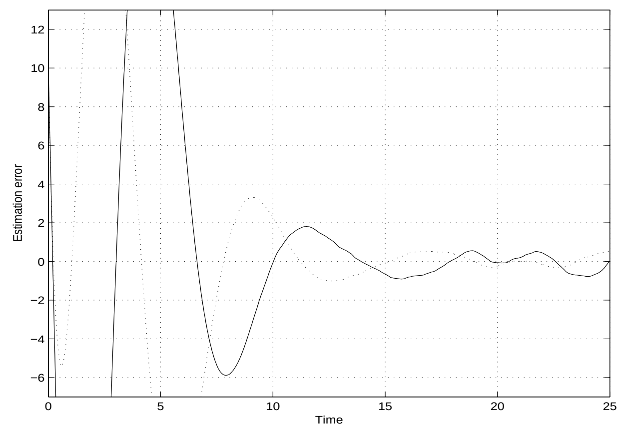

Fig. 2. Estimation error of Kalman filter

VI. Conclusions

FSN noise models are more practical than normal white noise models, since they allow the size (intensity) of the noises to be affinely related to the size (variance) of the signals they corrupt. Such noises are found in digital signal processing with both fixed and floating point arithmetic. Such models are found in analog sensors and actuators which produce more noise when the power supplies in these devices must provide more power (for an increased dynamic range of the signals in the estimation or control problem).

This paper derives the sufficient conditions for the existence of the state estimator with FSN noise models. By adding a mild constraint, the original problem (of estimating to within a specified covariance error bound), is solved as a convex problem. Associated with the solvable convex conditions, an LMI based approach is examined for the design of the estimator with the FSN model. This estimator design guarantees the performance requirement and the design algorithm is convergent.

VII. Acknowledgments

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VIII. References

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