# NETWORK GRAPH MODELS III

ADD. READING
CHIANG CH 9-10

**Random graph** E.R. 1959

- small diameter
- No structure (small clustering coeff.)
- Deg. distr. with exponential tail
  (All nodes have roughly same degree)

**Hybrid graph** W.S. 1998

- small diameter
- some structure (clustering coeff.)
- Deg. distr. exponential tail

These depend on $\beta$ but simulations show that very small "rewiring" suffices for decreasing diameter while keeping clustering.

We would like to have more variability in node degree distribution (Think about ISP vs. average user)

This is given by a power law degree distribution
For example PARETO DISTRIBUTION is defined by:

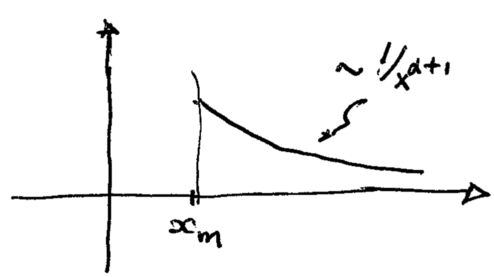
$$F_X(x) = \begin{cases} 1-\left(\frac{x_m}{x}\right)^{\alpha} & \text{for } x \geq x_m \\ 0 & \text{for } x < x_m \end{cases}$$

$$f_X(x) = \begin{cases} \dfrac{\alpha x_m^{\alpha}}{x^{\alpha+1}} & x \geq x_m \\ 0 & x < x_m \end{cases}$$

Notice that ~~...~~
$\alpha x_m^{\alpha}$ is a normalizing constant
so that

$$\alpha x_m^{\alpha} \int_{x_m}^{\infty} \frac{1}{x^{\alpha-1}}\, dx = \alpha x_m^{\alpha} \left[\frac{1}{\alpha x^{\alpha}}\right]_{x_m}^{\infty} = \frac{\alpha x_m^{\alpha}}{\alpha x_m^{\alpha}} = 1$$

and the pdf $\sim \frac{1}{x^{\alpha+1}}$




**EXERCISE**
Try to compute these results

$$E(X) = \begin{cases} \infty & \alpha \leq 1 \\ \dfrac{\alpha x_m}{\alpha-1} & \alpha > 1 \end{cases}$$

$$\mathrm{Var}\, X = \begin{cases} \infty & 1 < \alpha \leq 2 \\ \dfrac{x_m^2}{(\alpha-1)^2}\, \dfrac{\alpha}{\alpha-2} & \alpha > 2 \end{cases}$$

Discrete analogous to PARETO is ZIPF distr.

$$P(k) = \frac{1}{k^{\alpha}} \frac{1}{\sum_n \left(\frac{1}{n}\right)^{\alpha}}$$

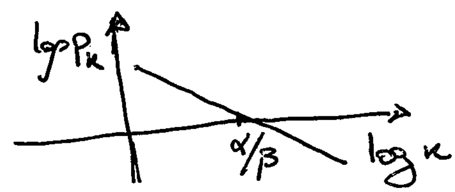
Again notice that $\frac{1}{\sum \left(\frac{1}{n}\right)^{\alpha}}$ is a normalizing constant so that

$$\sum_{k=m+1}^{N} P(k) = \frac{\sum (1/k)^{\alpha}}{\sum \left(\frac{1}{n}\right)^{\alpha}} = 1$$

$$P(k) = c\,k^{-\alpha}$$

$$\underbrace{\log P(k)}_{Y} = \underbrace{-\alpha \log k}_{-\alpha X} + \underbrace{\log c}_{\beta}$$

$$Y = -\alpha X + \beta$$




Power law distribution appears as a straight line with slope $-\alpha$ on a log-log plot.

In this case we have finite mean for $\alpha > 2$, finite variance for $\alpha > 3$ $\Rightarrow$ for $\boxed{2 < \alpha < 3}$ we can have a huge variance and a finite average. This means that for a finite node degree on average we have we still have some nodes that have a large number of connections. These will be the "outlier" nodes.

One popular model (that turns out to be wrong for Internet Modeling) is preferential attachment that exhibits

- small diameter
- clustering
- Power-law degree distr.

- START with connected graph of M nodes
- each new node connects with node $i$ with probability $P_i = \frac{k_i}{\sum_j k_j}$ where $k_i$ is node degree of $i$

With this model we have

$$P(k) \sim 1/k^3$$

$$\mathbb{E}(d) \sim \frac{\log N}{}$$

Preferential attachment leads to creation of "hubs" (nodes with large number connections) & Achille's heel as described in the book.
The book also explains how a little reflection shows that the "real internet" does not have these features.

A much better model (see additional reading handout) is

CONSTRAINED OPTIMIZATION NETWORK GROWTH

maximize $\sum_i x_i$ — sum-rates $\sum_i$ for end-to-end connection

s.t.

$\sum_i R_{ki} x_i \leq b_k \quad \forall k$ — sum-rate throug router k; $R_{ki}$ = portion of traffic of session i through router k

$x_i = \rho z_{S_i} z_{D_i}$ — rate for en-to-end connection i is proportional to demand at node $S_i$ and at destination node $D_i$

$b_k$ is bandwidth constraint of router k.

$\underline{x}$ is throughput vector that depends on routing matrix, that depends on graph.

Let P(G) the performance (maximized value of $\sum_i x_i$) it turns out that Preferential attachment has very small value of P(G).
There are other ways to obtain graph models that have power-law degree distributions & large values of P(G) obtained from constrained optimization. Example:

Node i attaches to node j

distance to other nodes

"benefit"

$$\min_{j<i} \alpha \cdot d_{ij} + h_j$$

$\boxed{\text{See also Handout for more}}$