ECE 158A MASSIMO FRANCESCHETTI

Lecture notes #3

What is bandwidth?

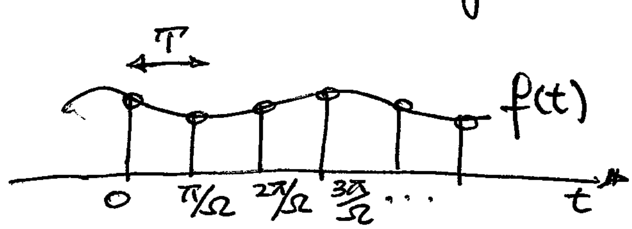
In lecture 2 we computed packet delay that depends on:
- link bandwidth (BIT-RATE)
- propagation delay (depends on distance ~ msec) Ex. $v = 10^8$ m/sec d = 10 Km ⇒ ΔT = 0.1 msec; 100 Km: 1 msec
- queueing delay (depends on load controlled by TCP)
- processing delay (Typically neglegible)

We said link bandwidth varies between ~100 Mbit/sec ~100 Gbit/sec
this is what impacts us most and drives Tech companies To update their networks.

But why is it called bandwidth?
How is frequency of a signal related to BITRATE?

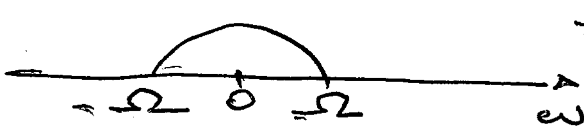
To explain this we have To go all the way down To physical layer and recall the SAMPLING THEOREM you have learned back in ECE45

For a bandlimited signal



$T = \frac{\pi}{\Omega}$

if I sample signal at least at twice the bandwidth I can reconstruct signal from its samples



$\frac{\Omega}{2\pi} = W \quad 2W = \frac{\Omega}{\pi}$

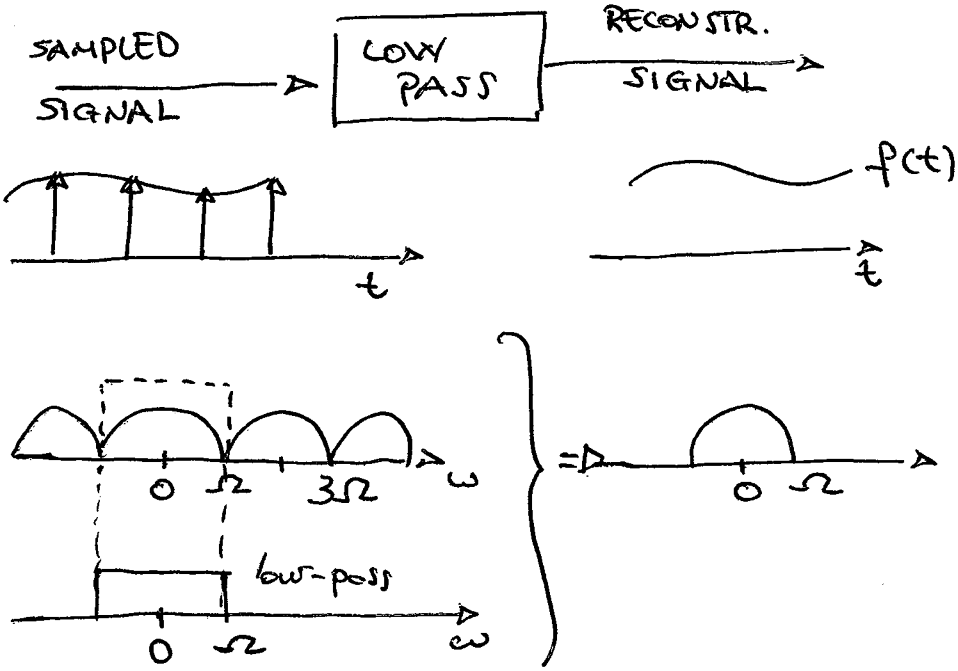
$T = \frac{1}{2W}$

This formula by Shannon-Whittaker-Kotelnikov

in Math known as "cardinal series" -

$$f(t) = \underbrace{\sum_n f\left(\frac{n\pi}{\Omega}\right)}_{\text{samples spaced by } T} \underbrace{\operatorname{sinc}(\Omega t - n\pi)}_{\text{reconstruction basis}} \qquad (1)$$

The reconstruction can be seen in the frequency domain as a low-pass filter applied to the replicated signal in frequency corresponding to a train of pulses weighted by sampling values



It follows that a signal of duration T_0 can be reconstructed from $\frac{T_0}{T}$ samples $\frac{T_0}{T} = \frac{T_0 \Omega}{\pi} = N_0$

N_0 is called Nyquist number

$\frac{N_0}{T_0} = \frac{\Omega}{\pi}$ is $\frac{\#\ \text{samples}}{\text{time}}$ identify a band limited signal

By transmitting a signal we can communicate $\frac{\Omega}{\pi}$ real numbers per unit time. So the rate in $\frac{\text{Reals}}{\text{sec}}$ is proportional to bandwidth, but why we talk of LINK BANDWIDTH in bits/sec ?

Can samples be arbitrary real numbers ?

$\boxed{\text{NO!}}$

First of all, there is an energy constraint.

Let $f(\frac{n\pi}{\Omega}) = x_n$ in Shannon formula. (1)
We have

$$f(t) = \sum_n x_n \operatorname{sinc}(\Omega t - n\pi)$$

With a computation that you should be able to do from ECE 45 we can show that energy is related to sampled values:

$$\int_{-\infty}^{+\infty} f^2(t)\,dt = \frac{\pi}{\Omega} \sum x_n^2$$

Hint: To perform the computation recall:

$$\frac{\Omega}{\pi} \operatorname{sinc}(\Omega t) \longleftrightarrow \operatorname{rect}\left(\frac{\omega}{2\Omega}\right)$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(\omega)|^2 d\omega = \int_{-\infty}^{+\infty} f^2(t)\,dt$$

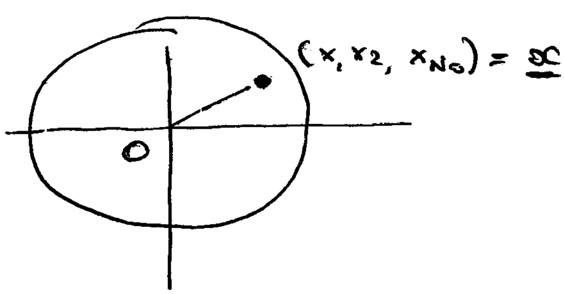
Now we impose an energy constraint

$$\int_{-\infty}^{+\infty} f^2(t)\,dt \leq P T_0$$

from which it follows a constraint on the samples:

$$\sum x_n^2 \leq \frac{P T_0 \Omega}{\pi} = P N_0$$

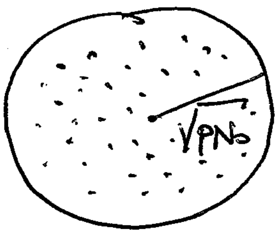
If we visualize $(x_1\, x_2 \quad x_{N_0})$ to be a point in a space of N_0 dimensions every signal sent corresponds to a point in this space and the energy constraints imposes all points to be inside a sphere of radius $\sqrt{PN_0}$



$$\text{distance}(\underline{x}, \underline{0}) = \sqrt{\sum_n x_n^2} \leq \sqrt{PN_0}$$

So samples cannot be arbitrary numbers, the energy constraint imposes signal points must be inside Ball!

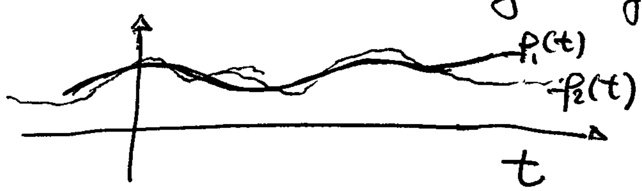
Still, we can pack ∞ many points in the ball,
this means ∞-many different signals of finite energy ~~[illegible]~~ can be represented with N_0 real numbers...



radius represents energy constraint
points represent signals
each point identified by N_0 coordinates
corresponding to samples of signal

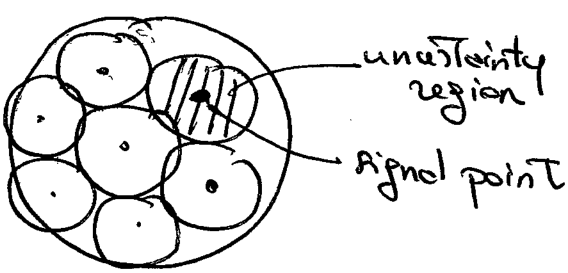
$\boxed{\text{NO!}}$ here is a

PROBLEM These points will be very close to each other!
This means nearby signals will be very similar



Hard to distinguish.

$\boxed{\text{Welcome Noise!}}$ The noise of the observation allows only $\boxed{M}$ signals to fit inside the ball and be distinguishable instead than ∞-many.



How to COMPUTE M

Radius of Ball $\sqrt{PN_0}$

Radius of Noise $\sqrt{N N_0}$

$$P = \frac{1}{T_0}\int_{-\infty}^{+\infty} f^2(t)\, dt \quad \text{Avg Power Signal}$$

$$N = \frac{1}{T_0}\int_{-\infty}^{+\infty} n^2(t)\, dt \quad \text{Avg Power noise}$$

If each signal has a range within ball of radius $\sqrt{PN_0}$
~~[illegible]~~ when it is observed it has a range within a ball of radius $\sqrt{N_0(P+N)}$

$$M = \frac{\text{Vol. Observed signal ball}}{\text{Vol Noise ball}} = \left(\frac{\sqrt{N_0(P+N)}}{\sqrt{N_0 N}}\right)^{N_0} = \left(\sqrt{1+\frac{P}{N}}\right)^{N_0}$$

Last step

Identify one out of M signals using $\log_2 M$ bits

$$\frac{1}{T_0}\log_2 M \text{ is } \frac{\#\text{bits}}{\text{time}} = \frac{N_0}{T_0}\log_2\left(\sqrt{1+\frac{P}{N}}\right) = \frac{\Omega \cancel{T_0}}{\cancel{T_0}\,\pi}\log\sqrt{1+\frac{P}{N}}$$

$$= \frac{\Omega}{2\pi}\log_2\left(1+\frac{P}{N}\right)$$

$$= \underbrace{W}_{\text{bandwidth in Hz}}\log_2\left(1+\underbrace{SNR}_{\text{Signal To noise Ratio}}\right)$$

For SNR = 1 which corresponds to 0dB: The capacity is W bits/sec. So 1MHz band is 1Mbs

---

EXAMPLE QUESTION: If we want a link of 20Mbs over a 3MHz band what is the SNR needed To support this rate?

$W = 3\cdot 10^6$ Hz

$C = 20\cdot 10^6$ bits/sec

$$20\cdot 10^6 = 3\cdot 10^6 \log(1+SNR)$$

$$\frac{20}{3} = \log(1+SNR)$$

$$102 \approx 1+SNR$$

$$101 = SNR$$