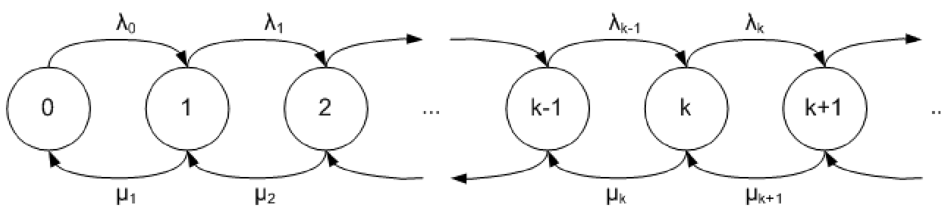
ECE 158A - MM1 queue

M/M/1 Queue

1) Consider the $M/M/1$ queue with arrival rate λ and departure (or service) rate μ. Let the *state* of the system be the number of packets in the system (queue and service), and let $s(t)$ denote the state at time t. The state transition diagram of this queue is represented in the figure below (where $\lambda = \lambda_1 = \lambda_2 = \cdots$ and $\mu = \mu_1 = \mu_2 = \cdots$). Remember that there is no upper bound to the queue size, and assume that $\lambda < \mu$. We want to compute the *stationary distribution* of the



queue size, that tells us how likely it is to be in each state when the queue has reached a stationary condition (read "after a large number of steps"). To be precise the stationary distribution is a vector

$$\pi = (\pi_0, \pi_1, \pi_2, \ldots),$$

such that

$$\Pr(s(t) = k) = \pi_k$$

for all $k \geq 0$ and for all t suitably large. As the values in π constitute a probability distribution, we have that

$$\sum_{k=0}^{\infty} \pi_k = 1.$$

As the diagram above suggests, we have that

$$\begin{aligned}
\lambda\pi_0 &= \mu\pi_1 \\
\lambda\pi_1 &= \mu\pi_2 \\
&\cdots \\
\lambda\pi_i &= \mu\pi_{i+1} \\
&\cdots
\end{aligned}$$

1) Use the fact above to express π_k, $k > 0$, as a function of π_0.

$$\pi_k = \left(\frac{\lambda}{\mu}\right)^k \pi_0$$

2) Using $\lambda < \mu$ and the fact that all π_k's sum to 1, compute π_0 (as a function of λ and μ).

$$1 = \sum_{k=0}^{\infty} \pi_k = \pi_0 \sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k = \pi_0 \frac{1}{1 - \lambda/\mu},$$

because the sum converges as $\lambda/\mu < 1$. Hence, we have

$$\pi_0 = 1 - \lambda/\mu.$$

3) Using the results above, compute the expected number of packets in the system at any given time. As you learnt in class, you should get $\frac{\lambda}{\mu-\lambda}$. You may find it useful that $\rho = \lambda/\mu < 1$.

Observe that $\pi_0 = 1 - \rho$. Then, the expected number of packets in the system is

$$\begin{aligned}
\sum_{k=0}^{\infty} k\pi_k &= \pi_0 \sum_{k=0}^{\infty} k\rho^k = rho(1-\rho) \sum_{k=0}^{\infty} k\rho^{k-1} \\
&= \rho(1-\rho) \sum_{k=0}^{\infty} \frac{\partial}{\partial \rho} \rho^k = \rho(1-\rho) \frac{\partial}{\partial \rho} \left(\sum_{k=0}^{\infty} \rho^k \right) \\
&= \rho(1-\rho) \frac{\partial}{\partial \rho} \frac{1}{1-\rho} = \frac{\rho}{1-\rho} \\
&= \frac{\lambda}{\mu - \lambda}.
\end{aligned}$$

Notice that the sum converges for $\rho < 1$, and this allowed to swap the derivative and the sum.

4)What is the expected time T_1 that a packet spends in the system (queue and service) if the arrival rate is λ and the departure rate is 3μ?

$$T_1 = \frac{1}{3\mu - \lambda}.$$