Lecture 2

ECE 278 Mathematics for MS Comp Exam

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Real Vector Spaces

- The rules for matrices are a special case of a **vector space**.
- The linear transformations described by finite-size matrices are a subset of all possible linear transformations defined in a **real n-dimensional vector space** $\mathbb{R}^n$.
- A **vector** in this space is simply an ordered n-tuple of real numbers.
- Vectors in $\mathbb{R}^2$ can be defined on a plane.
- Vectors in $\mathbb{R}^3$ are common in physics and computer graphics.
- **Signal space** to be studied later is a powerful vector space that enables a unified treatment of many mathematical concepts.

Complex Vector Spaces

- When the **vector** is an ordered n-tuple of complex numbers, then the space is a **complex vector space**
- Properties of vectors in vector spaces are a natural extension of the properties for numbers, which are called scalars in the context of vectors, (See Section 7.9 for a list axioms)
- A complex number is regarded as a scalar in a complex vector
- The same complex number can be regarded as a vector with two components (real and imaginary) in a real vector space

Linear Independence

- A linear combination of vectors $\mathbf{a}_i$ and scalars c_i can be written as

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_m\mathbf{a}_m$$

- The set of vectors $\{\mathbf{a}_i\}$ is **linearly independent** if the equation

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_m\mathbf{a}_m = \mathbf{0} \tag{1}$$

  implies that all the coefficients are zero where $\mathbf{0}$ is the zero vector.

- Otherwise if (1) also holds with some scalars not equal to zero, the vectors are **linearly dependent** with one or more vectors being a linear combination of the other vectors

- The vector space is **n-dimensional** if it contains n linearly-independent vectors

- That set is called a **basis**
  - there is **no requirement** that the set of linearly-independent vectors that form a basis is pairwise orthogonal—to be defined on the next slide

Real Inner Product Space

- For a real vector space $\mathbb{R}^n$ with the vectors regarded as column vectors the product

$$\mathbf{a}^T\mathbf{b}$$

  is called the **inner product**

- It is commonly written in several forms

$$\mathbf{a}^T\mathbf{b} = (\mathbf{a},\mathbf{b}) = \mathbf{a}\cdot\mathbf{b} = [a_1, \cdots, a_n]\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \sum_{i=1}^{n} a_1 b_1 + \cdots + a_n b_n$$

- Vectors whose inner is zero are called **orthogonal**
- The length or **norm** of a vector $\|\mathbf{a}\|$ is defined by

$$\|\mathbf{a}\| = \sqrt{\mathbf{a}\cdot\mathbf{a}}$$

- A vector of unit norm is called a **unit vector**

Complex Inner Product Space

- For a real vector space $\mathbb{C}^n$ with the vectors regarded as column vectors the product

$$\mathbf{b}^H \mathbf{a}$$

  is defined a the **inner product**

- Expressed as

$$[b_1^*, \cdots, b_n^*] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \sum_{i=1}^{n} a_1 b_1^* + \cdots + a_n b_n^*$$

- The **norm** is defined by

$$\|\mathbf{a}\| = \sqrt{|\mathbf{a}|^2}$$

  where

$$|\mathbf{a}|^2 = \mathbf{a}^H \mathbf{a} = \sum_{i=1}^{n} a_1 a_1^* + \cdots + a_n a_n^* = \sum_{i=1}^{n} |a|_i^2$$

- The term $|\mathbf{a}|^2$ is the **energy** in the vector.

Orthogonal and Orthonormal Basis

- Recall that any set of n linearly-independent vectors forms a basis for an n-dimensional vector space

  $$\mathbf{a}^T\mathbf{b}$$

  is called the **inner product**
- When b is a unit vector, the inner product is the **projection** of the vector $\mathbf{a}$ onto the vector $\mathbf{b}$
- An orthogonal basis is one for which the set of linearly-independent vectors are pairwise orthogonal.
- An orthonormal basis is one for which each basis vector has unit norm
- A set of n orthogonal vectors can be constructed from a set of n linearly-independent vectors using the ***Gram-Schmidt*** *procedure.*

Review Gram-Schmidt Procedure

- Start with one vector. Normalize.
- Project onto second vector. This value is the common part.
- Subtract the common part to produce a second orthogonal vector. Normalize.
- Repeat for each vector projecting out the parts that are common to the other vectors to produce orthogonal basis.

Vector Space of Matrices

- A set of linearly independent real two-by-two matrices form a four dimensional vector space.
- One **specific** orthonormal basis is

$$\mathbf{B}_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{B}_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \mathbf{B}_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \mathbf{B}_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

because **any** two-by-two matrix $\mathbf{A}$ can be written as a linear combination of the basis matrices weighted by scalars such that

$$\mathbf{A} = a_{11}\mathbf{B}_{11} + a_{12}\mathbf{B}_{12} + a_{21}\mathbf{B}_{21} + a_{22}\mathbf{B}_{22}$$

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Vector Space of Polynomials

- The set of all constant, linear and quadratic polynomials in x taken together forms a vector space of dimension 3 with a basis

$$\left[1, x, x^2\right]$$

- Note that this space is not $\mathbb{R}^3$
- Each basis "vector" is now considered a **function**—not a list of components
- If a vector space V contains a linearly-independent set of n vectors for every n no matter how large, then V is infinite-dimensional.
- **Signal space** is an infinite-dimensional vector space (either real or complex) that can express all continuous function on a fine interval $[a, b]$ as a linear combination of basis functions
- Common examples from other courses:
    - Fourier series
    - Sampling theorem

Inner Product and Norm of a Function

- The set of all complex-valued continuous functions on interval $0 \leq t \leq T$ is a complex vector space.
- On this space, we can define an inner product

$$\mathbf{a} \cdot \mathbf{b} \doteq \int_0^T a(t) b^*(t) \mathrm{d}t$$

- With the norm given by

$$\|\mathbf{a}\| = \sqrt{\int_0^T |a(t)|^2 \mathrm{d}t}$$

Example MS Math Problem (2016)

- Construct an orthonormal basis for the space over the interval $[0, 1]$ spanned by the functions: $x_1(t) = 1$, $x_2(t) = \sin(2\pi t)$, and $x_3(t) = \cos^2(2\pi t)$.
- **Solution**
  The basis functions can be determined using the *Gram-Schmidt procedure.*
- The function $x_1(t) = 1$ is already normalized so $\psi_1(t) = x_1(t) = 1$.
- Then project $x_2(t)$ onto $\psi_1(t)$

$$\int_0^1 (1)\sin(2\pi t)dt = 0$$

indicating that $x_2(t)$ is orthogonal to $\psi_1(t) = 1$.

- Normalizing $x_2(t)$ gives

$$\|x_2(t)\| = \sqrt{\int_0^1 \sin^2(2\pi t)dt} = \frac{1}{\sqrt{2}}$$

so that $\psi_2(t) = \sqrt{2}\sin(2\pi t)$.

Example MS Math Problem (2016)

- To determine the last basis function, express $\cos^2[2\pi t] = \frac{1}{2}(1+\cos[4\pi t])$.
- Projecting this function onto the other two functions gives

$$x_3(t)\cdot\psi_1(t) = \int_0^1 (1)\cdot\frac{1}{2}(1+\cos[4\pi t])\,dt = \frac{1}{2}$$

  indicating that the "DC" component of $x_3(t)$ is the same as $\psi_1(t)$.
- Repeating for $\psi_3(t)$, we have

$$x_3(t)\cdot\psi_2(t) = \int_0^1 \sqrt{2}\sin[2\pi t]\cdot\frac{1}{2}(1+\cos[4\pi t])\,dt = 0$$

  indicating that $x_3(t)$ is already orthogonal to $\psi_2(t)$.
- Thus the component of $x_3(t)$ that is orthogonal to both $\psi_1(t)$ and $\psi_2(t)$ is $\cos[4\pi t]$.
- Normalizing as before we have $\psi_3(t) = \sqrt{2}\cos(4\pi t)$.
- This completes the solution.