Lecture 3

ECE 278 Mathematics for MS Comp Exam

UCSD Jacobs | Electrical and Computer Engineering

Statement

- Discrete eigenvalue problem
$$\lambda \mathbf{x} = \mathbb{H}\mathbf{x} \tag{1}$$
where $\mathbb{H}$ is an n by n square matrix, $\mathbf{x}$ is a column vector of length n and λ is an **eigenvalue**
- Vector that satisfy 1 are called **eigenvectors**
- Continuous eigenvalue problem have
$$\lambda\psi(t) = \int_{-\infty}^{\infty} h(t,\tau)\psi(\tau)\mathrm{d}\tau$$
- Find eigenvalues λ_i and corresponding eigenvectors $\mathbf{x}_i$ (which are not the zero vector) is called an **eigenvalue problem**
- In terms of a transformation, an eigenvector is a vector at the output of the transformation that is in the same direction as the input vector.
  - It is only scaled by a number (which may be complex)

Finding Eigenvalues

- Rewrite

$$\lambda \mathbf{x} = \mathbb{H}\mathbf{x}$$

in terms of matrices so that

$$\left(\mathbb{H} - \lambda \mathbb{I}\right) \mathbf{x} = \mathbf{0} \tag{2}$$

- This is a homogeneous system of linear algebraic equations.
- This system of equation has a nontrivial solution when the determinant is equal to zero or

$$\begin{vmatrix} h_{11} - \lambda & h_{12} \\ h_{21} & h_{22} - \lambda \end{vmatrix} = 0.$$

- Apply rule for determinant of a two-by-two matrix gives

$$\left(h_{11} - \lambda\right)\left(h_{22} - \lambda\right) - h_{12}h_{21} = 0.$$

- Roots $\lambda_{1,2}$ to this quadratic equation are the eigenvalues
- In general finding the eigenvalues for a eigenproblem of order n requires finding the roots of an nth order polynomial.

Eigenvectors

- Each eigenvalue has a corresponding eigenvector
- To find eigenvector for eigenvalue λ_1, substitute eigenvalue into (2) to give two equations

$$\begin{aligned} (h_{11} - \lambda_1)\, x_1 + h_{12} x_2 &= 0 \\ h_{21} x_1 + (h_{22} - \lambda_1)\, x_2 &= 0 \end{aligned} \tag{3}$$

  where all the coefficients are known because $\mathbb{H}$ is known and λ_1 is known.

- Any solution to (3) for (x_1, x_2) from these equations gives an eigenvector up to a scalar multiple (which may be complex).

Example -Spring 2106 Exam

- Find the eigenvalues and eigenvectors of the following matrix

$$\begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}$$

- The characteristic determinant is

$$\begin{vmatrix} 3-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 2$$

- Setting this equation equal to zero, the eigenvalues are

$$3 \pm \sqrt{2}$$

Example -Spring 2106 Exam-2

- The first eigenvector is

$$\begin{array}{rr} \left(3-\left(3+\sqrt{2}\right)\right)x_1 & +x_2 = 0 \\ 2x_1 & 3-\left(3+\sqrt{2}\right)x_2 = 0 \end{array}$$

- Simplify

$$\begin{array}{rr} -\sqrt{2}x_1 & +x_2 = 0 \\ 2x_1 & -\sqrt{2}x_2 = 0 \end{array}$$

- Second equation is simply $-\sqrt{2}$ of first equation.
- For one solution set $x_2 = 1$ so that first eigenvector is

$$\begin{bmatrix} 1/\sqrt{2} \\ 1 \end{bmatrix}.$$

- Second eigenvector use $\lambda_2 = 3+\sqrt{2}$ so that

$$\begin{array}{rr} \sqrt{2}x_1 & +x_2 = 0 \\ 2x_1 & \sqrt{2}x_2 = 0 \end{array}$$

and

$$\begin{bmatrix} -1/\sqrt{2} \\ 1 \end{bmatrix}.$$

Example -Spring 2106 Exam-3

- **Write the vector $(1,1)$ as a linear combination of the eigenvectors.**
- Given the eigenvectors

$$\mathbf{e}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1 \end{bmatrix}$$

the equation is

$$a\mathbf{e}_1 + b\mathbf{e}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- This can be written as two equations:

$$\frac{a}{\sqrt{2}} - \frac{b}{\sqrt{2}} = 1$$

and

$$a + b = 1$$

- The solution is

$$\begin{aligned} a &= \frac{1}{2}(1+\sqrt{2}) \\ b &= \frac{1}{2}(1-\sqrt{2}) \end{aligned}$$

Multiple Eigenvalues

- Consider the matrix

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

- Using a cofactor expansion of the determinant (not covered here but required for exam - see Example 1 p. 294) the characteristic determinant is

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = -\lambda^3 - \lambda^2 + 21\lambda + 45$$

- Roots (eigenvalues) are $\lambda_1 = 5$, $\lambda_2 = -3$, $\lambda_3 = -3$.
  - Eigenvalues are not distinct
- Even though eigenvalues are not distinct, get three linearly-independent eigenvectors (see Example 2 p. 327)

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{x}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

Algebraic Multiplicity

- Let $\mathbb{C}$ be a $n \times n$ square matrix and let λ be an eigenvalue of $\mathbb{C}$.
- The **algebraic multiplicity** is the number of times λ is repeated which is called **the order** M_λ of the eigenvalue.
- The algebraic multiplicity is defined for each eigenvalue.
- For the previous example for $\lambda = -3$. Therefore M_λ is equal to two because it is a double root
- Sum of all the algebraic multiplicities for all the eigenvalues must equal n

Eigenspace and Geometric Multiplicity

- The **eigenspace** for a given eigenvalue is the null space of $\mathbb{C} - \lambda \mathbb{I}$
- For a given eigenvalue, the **null space** is the set of all column vectors $\{\mathbf{x}\}$ that satisfy

  $$(\mathbb{C} - \lambda \mathbb{I})\,\mathbf{x} = \mathbf{0}$$

  where $\mathbf{0}$ is the zero vector.
- Equivalently, the eigenspace is the union of the zero vector $\mathbf{0}$ and the set of the eigenvectors of λ.
- The **geometric multiplicity** m_λ is dimension of the eigenspace for each eigenvalue with algebraic multiplicity M_λ
- For the previous example, $m_\lambda = M_\lambda$ but this is not always true
- In general $m_\lambda \leq M_\lambda$ meaning that the geometric multiplicity is always less than or equal to the algebraic multiplicity
- If for every eigenvalue λ of the matrix $\mathbb{C}$ the geometric multiplicity is equal to the algebraic multiplicity, then matrix C is said to be **diagonalizable**.

Comments on Null Space and Rank of a Matrix

- Let $\mathbb{A}$ be an $n \times n$ square matrix.
- The **nullspace** is a subspace of the vector space $\mathbb{R}^n$.
- The dimension of the nullspace is called the **nullity** N of the matrix $\mathbb{A}$.
- **Rank-nullity theorem for a square matrix**
    - rank+nullity is the equal to the number or rows (or columns) in the square matrix A.
- For an $n \times n$ matrix, the nullity or dimension of its nullspace is $n - R$ where R is the rank of the matrix.

Example

- Let

$$\mathbb{C} = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ 6 & 0 & 2 \end{bmatrix}$$

- Then the eigenvalues are given by the solution to

$$\mathbb{C} - \lambda\mathbb{I} = \begin{bmatrix} 2-\lambda & 0 & 0 \\ 4 & 2-\lambda & 0 \\ 6 & 0 & 2-\lambda \end{bmatrix}$$

- Using Cramer's rule the determinant of this matrix can be written as

$$\det\left(\mathbb{C} - \lambda\mathbb{I}\right) = (2-\lambda)^2$$

- Therefore, there is a single eigenvalue $\lambda = 2$ with an algebraic multiplicity of 3.

- To find the geometric multiplicity, we need the eigenspace, which means solving the following equation

$$\left(\mathbb{C} - \lambda\mathbb{I}\right)\mathbf{x} = \mathbf{0}$$

  for $\lambda = 2$

Example-cont

- Let

$$\mathbb{C} - 2\mathbb{I} = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 6 & 0 & 0 \end{bmatrix} = \mathbf{0}$$

- Let R_i be the row number. Doing the following set of row operations: $R_2/4$, $R_3 - 6R_2$ and exchanging R_2 and R_1 gives the equivalent system

$$\mathbb{C} - \lambda\mathbb{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Therefore the solution to $(\mathbb{C} - 2\mathbb{I})\,\mathbf{x} = \mathbf{0}$ is equivalent to $x_1 = 0$, where x_1 is the first component of the vector $\mathbf{x}$.

- Clearly, the resulting matrix has a rank R equal to 1.
- Using the rank-nullity theorem gives the geometric multiplicity as

$$n - R = N$$
$$3 - 1 = 2$$

so that the geometric multiplicity for the eigenvalue $\lambda = 2$ is equal to two.

Example-cont

- The solution to $(\mathbb{C}-2\mathbb{I})\,\mathbf{x}=\mathbf{0}$ is equivalent to $x_1=0$, where x_1 is the first component of the vector $\mathbf{x}$.
- The other two components $x_2=s$ and $x_3=t$ are arbitary and are treated as free parameters so that the vector $\mathbf{x}$ can be written as

$$\mathbf{x}=\begin{bmatrix}0\\s\\t\end{bmatrix}=s\begin{bmatrix}0\\1\\0\end{bmatrix}+t\begin{bmatrix}0\\0\\1\end{bmatrix}$$

- Therefore, there are two basis vectors that span the eigenspace for $\lambda=2$ and thus the geometric multiplicity is equal to two.

Example Two

- Is the following matrix diagonalizable?

$$\begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

- This is equivalent to asking if the geometric multiplicity is equal to the algebraic multiplicity for every eigenvalue λ.
- The eigenvalues are the solution to

$$\det \begin{bmatrix} 2-\lambda & 2 & 2 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{bmatrix} = 0$$

or

$$(2-\lambda)^2 (3-\lambda) = 0$$

- Therefore the eigenvalue $\lambda = 2$ has an alegebraic multiplicity equal to two and the eigenvalue $\lambda = 3$ has an alegebraic multiplicity equal to one

Example Two- cont.

- Now form $(\mathbb{C} - \lambda\mathbb{I})$ for $\lambda = 2$. This gives

$$\mathbb{C} - 2\mathbb{I} = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

- Because one row is all zeros and the other two rows are a scalar multiplies of each other, the rank of this matrix is one.

- Using the rank-nullity formula, the nullity is equal to two and thus the geometric multiplicity is equal to the algebraic multiplicity for the eigenvalue $\lambda = 2$.

- Now form $(\mathbb{C} - \lambda\mathbb{I})$ for $\lambda = 3$. This gives

$$\mathbb{C} - 2\mathbb{I} = \begin{bmatrix} -1 & 2 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- Because the bottom two rows are a scalar multiplies of each other, the rank of this matrix is two and the geometric multiplicity is equal to one and is equal to the algebraic multiplicity.

- Therefore, because the geometric multiplicity is equal to the algebraic multiplicity for each eigenvalue, the matrix is diaogonalizable.