Lecture 4

ECE 278 Mathematics for MS Comp Exam

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- A discrete linear transformation is **self-adjoint** when the matrix representation $\mathbb{A}$ of that transformation is a square matrix that satisfies

$$\mathbb{A} = \mathbb{A}^H \tag{1}$$

(Book uses $\overline{\mathbb{A}}^T$ for $\mathbb{A}^H$)

- The matrix is called a **hermitian matrix**
  - elements a_{ij} of the matrix may be complex with $a_{ij} = a_{ji}^*$
  - have **real eigenvalues**
    - Converse is **not true**! (Matrices w/ real eigenvalues need not be hermitian.)
  - Distinct eigenvalues have **orthogonal eigenvectors**
- When the elements are real

$$\mathbb{A} = \mathbb{A}^T \tag{2}$$

- The matrix is called a **symmetric matrix**
  - elements a_{ij} of the matrix **must** be real with $a_{ij} = a_{ji}$
  - have **real eigenvalues** (Lie on the real line.)
  - Distinct eigenvalues have **orthogonal eigenvectors**

- A discrete linear transformation is **skew-hermitian** when the matrix representation $\mathbb{A}$ of that transformation is a square matrix that satisfies

$$-\mathbb{A} = \mathbb{A}^H \tag{3}$$

  - elements a_{ij} of the matrix may be complex with $a_{ij} = -a^*_{ji}$
  - **Eigenvalues** are **imaginary** or **zero**
  - Distinct eigenvalues have **orthogonal eigenvectors**

- When the elements are real

$$-\mathbb{A} = \mathbb{A}^T \tag{4}$$

  The matrix is called a **skew-symmetric matrix**

  - elements a_{ij} of the matrix **must** be real with $a_{ij} = -a_{ji}$
  - **Eigenvalues** are **imaginary** or **zero** (Lie on the imaginary line.)
  - Distinct eigenvalues have **orthogonal eigenvectors**

- A discrete linear transformation is **unitary** when the matrix representation $\mathbb{A}$ of that transformation is a square matrix that satisfies

$$\mathbb{A}^{-1} = \mathbb{A}^{H} \tag{5}$$

- When the elements are real

$$\mathbb{A}^{-1} = \mathbb{A}^{T} \tag{6}$$

  The matrix is called a **orthogonal matrix**

- These matrices preserve the **norm** or the inner product
- Eigenvalues are real or come in complex conjugate pairs with the magnitude of the eigenvalue equal to to one
  - All eigenvalues lie on the unit circle
- Determinant has a magnitude that is one

$$|\det \mathbb{A}| = 1$$

- For orthogonal matrix with real elements , this means that $\det \mathbb{A} = \pm 1$.
- Informally, they do not change the "length of the vector"
  - Note that the vector could be a function or matrix

- The matrix

$$\mathbb{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad (7)$$

is a **rotation matrix** in a plane in a counterclockwise direction by an angle θ
  - This is an orthogonal, skew-symmetric matrix
- The characteristic equation is

$$(\cos\theta - \lambda)(\cos\theta - \lambda) + \sin^2\theta = 0$$
$$\lambda^2 - 2\cos\theta\lambda + 1 = 0$$

- Using the quadratic formula gives

$$\begin{aligned} \lambda_{1,2} &= \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} \\ &= \cos\theta \pm \mathrm{i}\sin\theta \\ &= e^{\pm \mathrm{i}\theta} \end{aligned}$$

These are the eigenvalues. (Note unit magnitude and lie on unit circle.)

- The eigenvectors are

$$\begin{bmatrix} \cos\theta - (\cos\theta \pm \mathrm{i}\sin\theta) & -\sin\theta \\ \sin\theta & \cos\theta - (\cos\theta \pm \mathrm{i}\sin\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\to \sin\theta\,(\pm \mathrm{i}x + y = 0) \to v_{1,2} = \begin{bmatrix} \pm\mathrm{i} \\ 1 \end{bmatrix}$$

- As a check

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathrm{i} \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta + \mathrm{i}\cos\theta \\ \cos\theta + \mathrm{i}\sin\theta \end{bmatrix} = \lambda_1 \begin{bmatrix} \mathrm{i} \\ 1 \end{bmatrix}$$

- Generalizes transformations described by orthogonal matrices in a plane or in 3-space are **rotations**

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- The matrix

$$\mathbb{A} = \begin{bmatrix} 2 & -3i \\ 3i & 2 \end{bmatrix} \quad (8)$$

- The characteristic equation is

$$\begin{vmatrix} 2-\lambda & -3i \\ 3i & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda) - 9 = \lambda^2 - 4\lambda - 5 = 0$$

- Roots are $\lambda_1 = 5$ and $\lambda_2 = -1$

- Eigenvectors are solutions to

$$\begin{bmatrix} 2-\lambda_{1,2} & -3\mathrm{i} \\ 3\mathrm{i} & 2-\lambda_{1,2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0}$$

- For $\lambda_1 = 5$

$$\begin{bmatrix} -3 & -3\mathrm{i} \\ 3\mathrm{i} & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0}$$

- Second equation is i times first equation. Setting $y = 1$ gives

$$\mathbf{e}_1 = \begin{bmatrix} -\mathrm{i} \\ 1 \end{bmatrix}$$

- For $\lambda_1 = -1$

$$\begin{bmatrix} 3 & -3\mathrm{i} \\ 3\mathrm{i} & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0}$$

- Second equation is $-\mathrm{i}$ times first equation. Setting $y = 1$ gives

$$\mathbf{e}_2 = \begin{bmatrix} \mathrm{i} \\ 1 \end{bmatrix}$$

- Note $\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_1^H \cdot \mathbf{e}_2 = \begin{bmatrix} \mathrm{i} & 1 \end{bmatrix} \begin{bmatrix} \mathrm{i} \\ 1 \end{bmatrix} = 0$ so they are orthogonal.
- Normalizing by $1/\sqrt{2}$ makes them orthonormal.

- When the square matrix $\mathbb{A}$ has **distinct eigenvalues**, then the transformation described by $\mathbb{A}$ has a basis of the eigenvectors of $\mathbb{A}$
  - (not necessarily an orthogonal basis)
  - Can make it an orthonormal basis using Gram-Schmidt (See Lecture 3)
- A hermitian (symmetric) matrix has an **orthonormal basis** of eigenvector for $\mathbb{R}^n$
- This means that we can express any vector $\mathbf{x}$ in $\mathbb{R}^n$ as a superposition of the eigenvectors of the transformation that we are interested in
  - Linear time-invariant systems
  - Modes within electromagnetics

- Using the set of eigenvectors $\{\mathbf{e}_n\}$ of $\mathbb{R}^n$ as a basis, we can express any vector $\mathbf{x}$ in $\mathbb{R}^n$ as a superposition of the basis vectors as

$$\mathbf{x} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \ldots + c_n\mathbf{e}_n$$

- Apply the transformation described by $\mathbb{A}$ to the input vector $\mathbf{x}$

$$\begin{aligned}\mathbf{y} &= \mathbb{A}\mathbf{x} \\ &= \mathbb{A}\left(c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \ldots + c_n\mathbf{e}_n\right) \\ &= c_1\mathbb{A}\mathbf{e}_1 + c_2\mathbb{A}\mathbf{e}_2 + \ldots + c_n\mathbb{A}\mathbf{e}_n \\ &= c_1\lambda_1\mathbf{e}_1 + c_2\lambda_2\mathbf{e}_2 + \ldots + c_n\lambda_n\mathbf{e}_n\end{aligned}$$

- When input is expressed in terms of eigenvector of transformation, output of transformation simply scales each component by the corresponding eigenvalue.

- The is one of the fundamental methods of analysis for linear systems.

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- Previous lecture states that when the geometric multiplicity is equal to the algebraic multiplicity for every eigenvalue, the matrix is **diagonalizable**
- Restrict our discussion to **hermitian** matrices
  - Real eigenvalues and orthogonal eigenvectors
- Any hermitian matrix $\mathbb{A}$ can be diagonalized with a matrix $\mathbb{X}$ as follows

$$\mathbb{D} = \mathbb{X}^{-1}\mathbb{A}\mathbb{X}$$

  where the matrix $\mathbb{X}$ is formed from the orthogonal (column) eigenvectors of $\mathbb{A}$

- Diagonal elements are the eigenvalues of $\mathbb{A}$

- From earlier example, normalized eigenvectors are

$$\mathbf{e}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -\mathrm{i} \\ 1 \end{bmatrix} \qquad \mathbf{e}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathrm{i} \\ 1 \end{bmatrix}$$

- Therefore

$$\mathbb{X} = \frac{1}{\sqrt{2}} \begin{bmatrix} -\mathrm{i} & \mathrm{i} \\ 1 & 1 \end{bmatrix}$$

- This matrix is unitary so that $\mathbb{A}^{-1} = \mathbb{A}^H$ with

$$\mathbb{X}^{-1} = \mathbb{X}^H = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathrm{i} & 1 \\ -\mathrm{i} & 1 \end{bmatrix}$$

- Then

$$\begin{aligned} \mathbb{X}^{-1}\mathbb{A}\mathbb{X} &= \frac{1}{\sqrt{2}} \begin{bmatrix} \mathrm{i} & 1 \\ -\mathrm{i} & 1 \end{bmatrix} \begin{bmatrix} 2 & -3\mathrm{i} \\ 3\mathrm{i} & 4 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -\mathrm{i} & \mathrm{i} \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} \mathrm{i} & 1 \\ -\mathrm{i} & 1 \end{bmatrix} \begin{bmatrix} -5\mathrm{i} & -\mathrm{i} \\ 5 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

- A transformation described by a matrix need not have real eigenvalues and the eigenvectors need not be orthogonal as would be the case for a hermitian matrix.
- A useful decomposition of the matrix $\mathbb{A}$, called the *singular-value decomposition*, is

$$\mathbb{A} \quad = \quad \mathbb{U}\,\mathbb{M}\,\mathbb{V}^H. \tag{9}$$

- The matrices $\mathbb{U}$ and $\mathbb{V}$ are each unitary.
- The columns of $\mathbb{U}$ are the eigenvectors of $\mathbb{A}\mathbb{A}^H$
- The columns of $\mathbb{V}$ are the eigenvectors of $\mathbb{A}^H\mathbb{A}$.
- The only nonzero elements of the matrix $\mathbb{M}$ are on the diagonal, whose elements are denoted m_k.
- These elements are called the *singular values* of $\mathbb{A}$.
- They are the nonnegative square roots $\sqrt{\xi_k}$ of the eigenvalues ξ_k of the real symmetric matrix $\mathbb{A}\mathbb{A}^H$ so that $\xi_k = |m_k|^2$.

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- When $\mathbb{A}$ describes a transformation of the amplitude of a signal, the real symmetric matrix $\mathbb{A}\mathbb{A}^H$ describes the transformation of the signal power.