Lecture 6

ECE 278 Mathematics for MS Comp Exam

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- The set of all complex exponentials $e^{i2\pi ft}$ whose frequency f is an integer multiple of $1/T$ is a basis $\{\psi_n(t)\}$ for signal space on the interval $[0,T]$.
- A function in this signal space can be expanded in a Fourier series using the Fourier basis $\{\psi_n(t)\} = \{e^{in2\pi ft}\}$.
- The Fourier coefficients s_n are the expansion coefficients in the Fourier basis.
- The expansion of an arbitrary function $s(t)$ on the interval $[0,T]$ is then

$$s(t) = \; = \sum_{n=-\infty}^{\infty} s_n \psi_n(t) \tag{1}$$

$$= \sum_{n=-\infty}^{\infty} s_n e^{in2\pi t/T} \tag{2}$$

where

$$s_n = \int_0^T s(t) e^{-in2\pi t/T} \mathrm{d}t$$

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- The expression for the expansion coefficient s_n

$$s_n = \int_0^T s(t) e^{-\mathrm{i}n2\pi t/T}$$

is in the form of an inner product with

$$s_n = s(t) \cdot \psi_n^*(t)$$

where

$$\psi_n(t) = e^{\mathrm{i}n2\pi t/T}$$

is a Fourier series basis function.

- Any two basis functions are orthonormal because

$$\begin{aligned}\psi_n(t) \cdot \psi_m(t) &= \int_0^T e^{-\mathrm{i}n2\pi t/T} e^{\mathrm{i}m2\pi t/T} \mathrm{d}t \\ &= \int_0^T e^{\mathrm{i}(m-n)2\pi t/T} \mathrm{d}t \\ &= \delta_{mn}\end{aligned}$$

where δ_{mn} is the Kronecker impulse.

Resonse of an LTI system to a Fourier basis function (which is eigenfunction of LTI system) is

$$e^{in2\pi t/T} \longrightarrow \boxed{H(f)} \longrightarrow H(n2\pi/T) \times e^{in2\pi t/T}$$

Eigenfunction — Transfer Function — Eigenvalue — Eigenfunction

- Given a system described by the transfer function $H(f)$ the output is

$$y(t) = \sum_{n=-\infty}^{\infty} \underbrace{s_n}_{\text{coeff.}} \underbrace{H(n2\pi/T)}_{\text{eigenvalue}} \underbrace{e^{in2\pi t/T}}_{\text{eigenfunction}}$$

- Note that the Fourier basis is only **one** of an infinite number of orthogonal functions that can be used as a basis on a finite time interval
  - Goal is to chose the basis such that the basis functions are the eigenfunctions of the system!

- Example: **Fourier-Legendre Series**

$$\psi_n(t) = P_n(x)$$

where (See Section 5.2 for definition with m and n interchanged)

$$P_m(x) = \sum_{n=0}^{\infty} \underbrace{(-1)^n \frac{(2m-2n)!}{2^m n!(m-n)!(m-2n)!}}_{\text{polynomial coefficient}} x^{m-2n}$$

- Coefficient of the expansion is

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x) \mathrm{d}x$$

where term $(2m+1)/2$ normalizes the expression because

$$\|P_m(x)\| = \sqrt{\int_{-1}^{1} P_m(x) \mathrm{d}x} = \sqrt{\frac{2}{2m+1}}$$

- A deterministic baseband waveform $s(t)$ whose spectrum $S(f)$ is zero for $|f|$ larger than $\mathcal{W}$ is called a **bandlimited waveform**.
- Because $(-\mathcal{W}, \mathcal{W})$ defines an interval on the frequency axis, the set of functions on this interval is a signal space and is spanned by a countable set of basis functions.
- One such set of basis functions is the Nyquist basis $\{\mathrm{sinc}\big((t-T)/T\big)\}$ as is given by the Nyquist-Shannon sampling theorem.

- The sampling theorem can be described by setting $\mathcal{W} = 1/2$ (or $T = 1$) and multiplying $s(t)$ by $\mathrm{comb}(t)$, which is defined as

$$\mathrm{comb}(t) \doteq \sum_{j=-\infty}^{\infty} \delta(t-j).$$

- This produces a sampled waveform with the samples $s(j)$ spaced by the sampling interval $T_s = 1$ with the "expansion coefficient" $s_j \doteq s(j)$ given as

$$s(j) = s(t)\mathrm{comb}(t) = s(t) \sum_{j=-\infty}^{\infty} \delta(t-j)$$

- We will show that any bandlimited function $s(t)$ has the following expansion

$$s(t) = \sum_{j=-\infty}^{\infty} s_j \mathrm{sinc}\Big(t-j\Big)$$

where the set $\Big\{\mathrm{sinc}\Big(t-j\Big)\Big\}$ of shifted sinc functions forms an orthogonal basis

Two sampled waveforms are shown in Figure below for two different sampling rates.

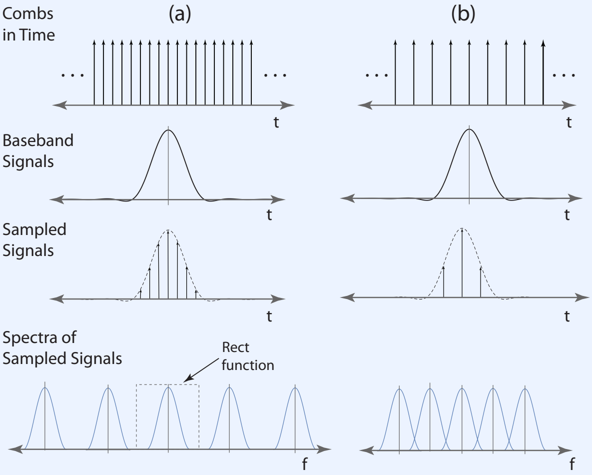

Figure: (a) Sampling at greater than the Nyquist rate. (b) Sampling at less than the Nyquist rate showing the effect of aliasing.

- Use the Fourier transform pair

$$\sum_{j=-\infty}^{\infty} \delta(t-j) \longleftrightarrow \sum_{j=-\infty}^{\infty} \delta(f-j)$$

or

$$\text{comb}(t) \longleftrightarrow \text{comb}(f).$$

- Now use the convolution property of the Fourier transform

$$s(t)h(t) \longleftrightarrow S(f) \circledast H(f)$$

with the dual property

$$s(t) \circledast h(t) \longleftrightarrow S(f)H(f).$$

- Apply ths property to $s(j) = s(t)\text{comb}(t)$ to give

$$s(t)\text{comb}(t) \longleftrightarrow S(f) \circledast \text{comb}(f)$$
$$\Big(s(t)\text{comb}(t)\Big) \circledast \text{sinc}(t) \longleftrightarrow \Big(S(f) \circledast \text{comb}(f)\Big)\text{rect}(f)$$

where the Fourier transform pairs

$$\text{rect}(t) \longleftrightarrow \text{sinc}(f)$$
$$\text{sinc}(t) \longleftrightarrow \text{rect}(f)$$

have been used.

$$s(t)\text{comb}(t) \longleftrightarrow S(f) \circledast \text{comb}(f) \quad (3)$$

$$\big(s(t)\text{comb}(t)\big) \circledast \text{sinc}(t) \longleftrightarrow \big(S(f) \circledast \text{comb}(f)\big)\text{rect}(f) \quad (4)$$

- The left side of (3) is an infinite sequence of impulses with the area of the kth impulse equal to the kth sample value.
- The right side is the spectrum of the sampled waveform $S(f) \circledast \text{comb}(f)$ and is shown at the bottom of Figure 1 for two different sampling rates.
- For the left set of curves in Figure 1a, multiplying the right by $\text{rect}(f)$, which is shown as a dashed line, recovers $S(f)$ because the **support** of $S(f)$ is $[-1/2, 1/2]$.
- For this finite support, the images of the original spectrum $S(f)$ do not overlap in $S(f) \circledast \text{comb}(f)$.

- Multiplication by $\text{rect}(f)$ in frequency corresponds to a convolution in time with $\text{sinc}(t)$.
- Because $[S(f) \circledast \text{comb}(f)]\text{rect}(f) = S(f)$, the convolution of $\text{sinc}(t)$ with the left side of (4) recovers $s(t)$ so that

$$\begin{aligned} s(t) &= \text{sinc}(t) \circledast \left[s(t)\text{comb}(t)\right] \\ &= \sum_{j=-\infty}^{\infty} s(j)\text{sinc}\big(t-j\big) \qquad (5) \\ &= \sum_{j=-\infty}^{\infty} s_j \psi_j(t) \qquad (6) \end{aligned}$$

where $\psi_j(t) = \text{sinc}\big(t-j\big)$ are the basis functions and $s_j = s(j)$ is the coefficient of the expansion.

- A waveform $s(t)$ bandlimited to $[-1/2, 1/2]$ can be expanded using $\{\text{sinc}(t-j)\}$ with the coefficients simply being samples $s(j)$.

Orthogonality of Shifted Nyquist Pulses

- Shifted sinc pulses are orthogonal because

$$\psi_n(t) \cdot \psi_m(t) = \int_{-\infty}^{\infty} \mathrm{sinc}\big(t-n\big)\mathrm{sinc}\big(t-m\big)\mathrm{d}t$$
$$= \delta_{mn}$$

- For arbitrary sampling interval T_s , applying the scaling property of the Fourier transform gives

$$s(t) = \sum_{j=-\infty}^{\infty} s_j \psi_j(t) \quad (7)$$
$$\sum_{j=-\infty}^{\infty} \underbrace{s(jT_s)}_{\text{coefficient}} \; \underbrace{\mathrm{sinc}\big(2\mathcal{W}t - j\big)}_{\text{basis function}}, \quad (8)$$

where $T_s = 1/2\mathcal{W}$.

- In this way, the sequence of sinc functions is a sequence of **interpolating functions** for a bandlimited signal.

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- The images of the original signal spectrum $S(f)$ shown in Figure 1 are offset by the sampling rate $R_s = 1/T_s$.
- When $R_s \geq 2\mathcal{W}$, the images do not overlap. The minimum sampling rate $R_s = 2\mathcal{W}$ is called the *Nyquist rate*.
- When R_s is greater than or equal to the Nyquist rate, the images do not overlap and the original signal $s(t)$ can be reconstructed as given by (7).
- When the sampling rate is smaller than the Nyquist rate, the images of the original signal spectrum $S(f)$ overlap.
- This effect is called ***aliasing***
- Aliasing is a form of signal distortion that replicates frequency components in the original signal at other frequencies in the reconstructed signal.

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- A Nyquist pulse is any pulse that is zero at all nonzero integers and has the value one when its argument is zero.

- One class of Nyquist pulses with various bandwidths and timewidths is the set of pulses with *raised cosine spectra* given by

$$q(t) = \frac{\sin(\pi t)\cos(\beta \pi t)}{\pi t\left(1-(2\beta t)^2\right)}, \tag{9}$$

  where β is a parameter in the range $[0, 1]$ that controls the temporal duration of the pulse. (β=0 gives a sinc pulse)

- For large t and a nonzero value of β, this pulse $q(t)$ eventually decays as t^{-3}. The spectrum $Q(f)$ is

$$Q(f) = \begin{cases} 1 & \text{for} \quad |f| \le (1-\beta)/2 \\ \frac{1}{2}\left(1+\cos\left[\frac{\pi}{\beta}\left(|f|-(1-\beta)/2\right)\right]\right) & \text{for} \quad (1-\beta)/2 \le |f| \le (1 \\ 0 & \text{otherwise} \end{cases}$$

  The pulse has a two-sided total bandwidth $(1+\beta)$ in constrast to 1 for a sinc pulse.

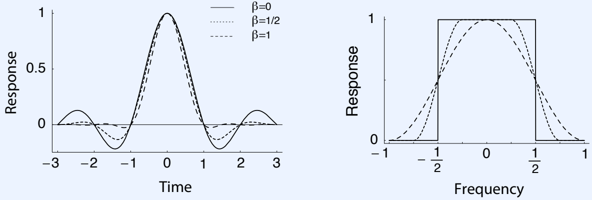

Figure: Time and frequency plots of a raised-cosine spectrum Nyquist pulse as a function of β.

- *Shifted raised cosine spectra* Nyquist pulses also form an orthogonal basis that can be used to represent a signal.
- Choice depends bandwidth and computational complexity.
- A sinc pulse has the smallest bandwidth of any basis function that can be used to represent the signal.
- However, expanding the signal using sinc pulses is computationally intensive because $\text{sinc}(t)$ decays slowly as $1/t$.
- A large number of terms must be summed to accurately represent the signal in the sinc expansion basis.
- Noting that the sum of $1/n$ over n is divergent also alerts us to the concern expressing the function using sincs is quite sensitive to instability or amplitude saturation.
- For other raised cosine pulses with $\beta \neq 0$, and large t the pulse decays as t^{-3} and thus it is easier to computationally represent this kind of basis function.
- The cost is a larger bandwidth.