Lecture 12

Complex Circularly-Symmetric Gaussians
- Autocovariance
- Magnitude/Phase Representation
- Marginal Phase Distribution

Poisson Count Process
- Probability Mass Function
- Mean and Variance
- Sum of Two Poissons
- Waiting Time

# Lecture 12

## ECE 278 Mathematics for MS Comp Exam

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Complex Circularly-Symmetric Gaussians
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- A *complex gaussian random variable* $\underline{z} = \underline{x} + \mathrm{i}\underline{y}$ has components $\underline{x}$ and $\underline{y}$ described by a real bivariate gaussian random variable $(\underline{x}, \underline{y})$.
- A *complex gaussian random vector* denoted as $\underline{\mathbf{z}} = \underline{\mathbf{x}} + \mathrm{i}\underline{\mathbf{y}}$ has components $\underline{z}_k$ described by a real bivariate gaussian random variable $(\underline{x}_k, \underline{y}_k)$.
- A complex gaussian random variable $\underline{z} = \underline{x} + \mathrm{i}\underline{y}$ with independent, zero-mean components $\underline{x}$ and $\underline{y}$ of equal variance is a circularly-symmetric gaussian random variable.

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- The corresponding probability density function is called a complex circularly-symmetric gaussian probability density function.
- A complex circularly-symmetric gaussian random variable has the property that $e^{\mathrm{i}\theta}\underline{z}$ has the same probability density function for all $\theta$.
- Generalizing, a complex, jointly gaussian random vector $\underline{\mathbf{z}} = \underline{\mathbf{x}} + \mathrm{i}\underline{\mathbf{y}}$ is circularly symmetric when the vector $e^{\mathrm{i}\theta}\underline{\mathbf{z}}$ has the same multivariate probability density function for all $\theta$.
- Such a probability density function must have a zero mean.

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- The multivariate probability density function for a complex circularly-symmetric gaussian random vector $\underline{\mathbf{z}}$ is

$$f_{\underline{\mathbf{z}}}(\mathbf{z}) = \frac{1}{\pi^N \det \mathbb{W}} e^{-\mathbf{z}^H \mathbb{W}^{-1} \mathbf{z}}. \quad (1)$$

- Using the properties of determinants, the leading term $(\pi^N \det \mathbb{W})^{-1}$ in (1) can be written as $\det(\pi \mathbb{W})^{-1}$.
- The term $\mathbb{W}$ in (1) is the autocovariance matrix given by

$$\mathbb{W} \doteq \langle \underline{\mathbf{z}}\underline{\mathbf{z}}^H \rangle \quad (2)$$

with $^H$ denoting the complex conjugate transpose.

- Because $\underline{\mathbf{z}}$ is complex, this matrix is hermitian.

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- The ***complex covariance matrix*** for this block, can be written as

$$\mathbb{W}_{\text{noise}} = \langle \underline{\mathbf{n}}\,\underline{\mathbf{n}}^H \rangle = 2\sigma^2 \mathbb{I}, \tag{3}$$

where $\mathbb{I}$ is the $M$ by $M$ identity matrix.

- The corresponding probability density function is

$$f(\mathbf{n}) = \frac{1}{(2\pi\sigma^2)^M} e^{-|\mathbf{n}|^2/2\sigma^2}, \tag{4}$$

where $|\mathbf{n}|^2 = \sum_{k=1}^{M} |n_k|^2 = \sum_{k=1}^{M} (x_k^2 + y_k^2)$ is the squared euclidean length of the random complex vector $\underline{\mathbf{n}}$.

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- Start with components are independent with the corresponding bivariate gaussian density function given by

$$\begin{aligned} f(x,y) &= f_{\underline{x}}(x) f_{\underline{y}}(y) \\ &= \left(\frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}\right)\left(\frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2}\right). \end{aligned} \tag{5}$$

- The two marginal probability density functions are both gaussian with the same variance.
- The two samples described by this joint distribution are independent.

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- For a magnitude/phase representation of a circularly-symmetric gaussian, the probability density function of the magnitude $f(A)$ can be obtained by a change of variables using $P = A^2/2$ and $\mathrm{d}P/\mathrm{d}A = A$, where the factor of $1/2$ accounts for the passband signal.
- This variable transformation yields

$$f(A) = \frac{A}{\sigma^2} e^{-A^2/2\sigma^2} \qquad \text{for } A \geq 0. \tag{6}$$

- This is the distribution $f(A)$ for the magnitude
- It is a Rayleigh probability density function

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- .Similarly, the probability density function $f(\phi)$ for the phase is a uniform probability density function $f(\phi) = (1/2\pi)\text{rect}(\phi/2\pi)$.
- The joint probability density function of these two independent (polar) components is

$$f(A,\phi) = f_{\underline{A}}(A) f_{\underline{\phi}}(\phi) = \frac{A}{2\pi\sigma^2} e^{-A^2/2\sigma^2} \quad \text{for } A \geq 0. \tag{7}$$

- Transforming from polar coordinates to cartesian coordinates recovers (5).

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- A Poisson random counting process is shown in Figure 1b.
- It is characterized by a ***arrival rate*** $\mu(t)$.

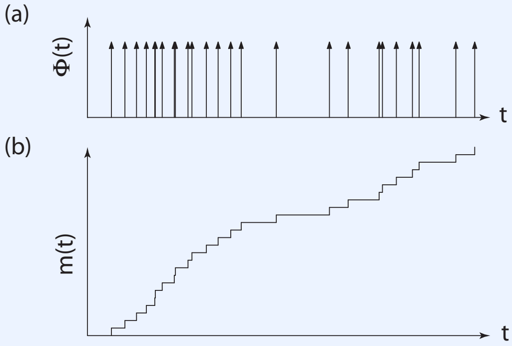


Figure: (a) A realization of a random photoelectron arrival process g$(t)$. (b) The integral of g$(t)$ generates the Poisson counting process m$(t)$.

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- A distinguishing feature of a Poisson counting process is that the number of counts in two nonoverlapping intervals are statistically independent for any size or location of the two intervals.
- This property is called the *independent-increment property* .

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- The probability $p_d$ of generating an event in subinterval $\Delta t$ is proportional to the product of a ***arrival rate*** $\mu$ and the interval $\Delta t$.
- The subinterval $\Delta t$ can be chosen small enough so that the probability of generating one counting event within $\Delta t$ is, to within order $\Delta t$, given by

$$p_d = \mu \Delta t = \frac{\mu T}{\mathsf{M}} = \frac{\mathsf{W}}{\mathsf{M}}, \tag{8}$$

where $\mathsf{W} = \mu T$ is the mean number of counts over an interval $T = M\Delta t$.

- The probability that no counts are generated within an interval of duration $\Delta t$ is, to order $\Delta t$, $1 - \mu \Delta t$.

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- The probability of generating m independent counts within a time $T = M\Delta t$ is given by a ***binomial probability mass function***

$$p(\mathsf{m}) = \frac{\mathsf{M}!}{\mathsf{m}!(\mathsf{M}-\mathsf{m})!}(p_d)^{\mathsf{m}}(1-p_d)^{\mathsf{M}-\mathsf{m}} \qquad \text{for } \mathsf{m} = 0, 1, 2, \dots, \mathsf{M}.$$

- Substituting $p_d = \mathsf{W}/\mathsf{M}$ from (8) yields

$$\begin{aligned} p(\mathsf{m}) &= \frac{\mathsf{M}!}{\mathsf{M}^{\mathsf{m}}(\mathsf{M}-\mathsf{m})!}\frac{\mathsf{W}^{\mathsf{m}}}{\mathsf{m}!}\left(1-\frac{\mathsf{W}}{\mathsf{M}}\right)^{\mathsf{M}-\mathsf{m}} \\ &= \frac{\mathsf{M}(\mathsf{M}-1)\dots(\mathsf{M}-\mathsf{m}+1)}{\mathsf{M}^{\mathsf{m}}}\frac{\mathsf{W}^{\mathsf{m}}}{\mathsf{m}!}\left(1-\frac{\mathsf{W}}{\mathsf{M}}\right)^{\mathsf{M}-\mathsf{m}}. \qquad (9) \end{aligned}$$

- Referring to (8), if $\mu$ and $T$ are both held fixed, then W is constant.

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- Therefore, in the limit as $\mathsf{W}/\mathsf{M} = \mu\Delta t$ goes to zero, M goes to infinity.
- The first term in (9) approaches one because the numerator approaches $\mathsf{M}^{\mathsf{m}}$.
- In the last term, the finite value of m relative to the value of M can be neglected.
- This produces $(1 - \mathsf{W}/\mathsf{M})^{\mathsf{M}}$ which goes to $e^{-\mathsf{W}}$ as M goes to infinity.
- Therefore, the probability mass function of the number of counts generated over an interval $T$ is

$$p(\mathsf{m}) = \frac{\mathsf{W}^{\mathsf{m}}}{\mathsf{m}!} e^{-\mathsf{W}} \qquad \text{for } \mathsf{m} = 0, 1, 2, \ldots \tag{10}$$

which is the ***Poisson probability distribution*** (or the Poisson probability mass function) with the mean $\langle \underline{\mathsf{m}} \rangle$ given by $\mu T = \mathsf{W}$.

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- The variance of the Poisson probability distribution can be determined using the characteristic function

$$C_{\mathsf{m}}(\omega) = \sum_{\mathsf{m}=0}^{\infty} e^{i\omega\mathsf{m}} p(\mathsf{m}) = \sum_{\mathsf{m}=0}^{\infty} e^{i\omega\mathsf{m}} \frac{\mathsf{W}^{\mathsf{m}}}{\mathsf{m}!} e^{-\mathsf{W}}$$

- The summation has the form $\sum_{\mathsf{m}=0}^{\infty} \frac{1}{\mathsf{m}!} x^{\mathsf{m}} = e^{x}$ with $x = \mathsf{W}e^{i\omega}$. Then $C_{\mathsf{m}}(\omega)$ reduces as

$$C_{\mathsf{m}}(\omega) = e^{\mathsf{W}(e^{i\omega}-1)}. \tag{11}$$

# Mean and Variance-2

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- From Lecture 12, the mean-squared value in terms of the characteristic function is

$$\begin{aligned} \langle \underline{m}^2 \rangle &= \left| \frac{1}{i^2} \frac{d^2}{d\omega^2} C_m(\omega) \right|_{\omega=0} \\ &= e^{W(e^{i\omega}-1)} \left( We^{i\omega} + \left( We^{i\omega} \right)^2 \right) \Big|_{\omega=0} \\ &= W + W^2. \end{aligned} \tag{12}$$

- Accordingly

$$\sigma_m^2 = \langle \underline{m}^2 \rangle - \langle \underline{m} \rangle^2 = W \tag{13}$$

is the variance of the Poisson distribution.

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- Expression (13) shows that the variance $\sigma^2_{\mathsf{m}}$ is equal to the mean.
- A random variable described by the Poisson probability distribution is called a ***Poisson random variable***.
- The sum of two independent Poisson random variables $\underline{\mathsf{m}}_1$ and $\underline{\mathsf{m}}_2$ is a Poisson random variable $\underline{\mathsf{m}}_3$.
- Let $p_1(\mathsf{m})$ and $p_2(\mathsf{m})$ be two Poisson probability distributions with mean values $\mathsf{W}_1$ and $\mathsf{W}_2$ respectively. Then the probability distribution $p_3(\mathsf{m})$ for $\underline{\mathsf{m}}_3$ is the convolution $p_3(\mathsf{m}) = p_1(\mathsf{m}) \circledast p_2(\mathsf{m})$.

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- The convolution property of a Fourier transform states that the two characteristic functions satisfy

$$C_3(\omega) \quad = \quad C_1(\omega)C_2(\omega),$$

where $C_i(\omega)$ is the characteristic function of $p_i(\mathrm{m})$.

- Substitute $C_1(\omega) = e^{\mathrm{W}_1(\mathrm{e}^{\mathrm{i!}}-1)}$ and $C_2(\omega) = e^{\mathrm{W}_2(e^{\mathrm{i}\omega}-1)}$ on the right (cf. (11)) and take the inverse Fourier transform to give

-

$$p_3(\mathrm{m}) \quad = \quad \frac{(\mathrm{W}_1 + \mathrm{W}_2)^{\mathrm{m}}}{\mathrm{m}!} e^{-(\mathrm{W}_1+\mathrm{W}_2)} \qquad \text{for m} = 0, 1, 2, \ldots . \quad (14)$$

- Accordingly, the sum of two independent Poisson random variables with means $\mathrm{W}_1$ and $\mathrm{W}_2$ respectively, is a Poisson random variable with a mean $\mathrm{W}_1 + \mathrm{W}_2$.

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- The probability density function for the waiting time $t_1$ to observe one event of a Poisson process from any starting time is denoted $f(t_1)$.
- For a Poisson stream, this probability density function does not depend on the starting time.
- For a constant arrival rate $\mu$, the cumulative probability density function of the random arrival time $\underline{t}_1$ is equal to $1 - p_0$ where $p_0$ is the probability that no counts are generated over the interval $t_1$.

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- The probability $p_0$ is determined using the Poisson probability distribution defined in (10) with W $= \mu t_1$ and m $= 0$. This yields $p_0 = e^{-\mu t_1}$.
- The cumulative probability density function is

$$F(t_1) = 1 - e^{-\mu t_1} \qquad \text{for } t_1 \geq 0.$$

- The corresponding probability density function is determined using (see Lecture 11)

$$\begin{aligned} f(t_1) &= \frac{\mathrm{d}}{\mathrm{d}t} F(t_1) \\ &= \mu e^{-\mu t_1} \qquad \text{for } t_1 \geq 0, \end{aligned} \tag{15}$$

which is an exponential probability density function with mean $\mu^{-1}$ and variance $\mu^{-2}$.

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- Given that the count events are independent, the waiting time for the second photoelectron event, and each subsequent event, is an independent exponential probability density function.
- The probability density function of the waiting time to generate $k$ photoelectrons is the sum of $k$ independent, exponentially distributed random variables—one random variable for each photoelectron arrival, and each with the same expected value $\mu^{-1}$.
- This is a ***gamma probability density function*** with parameters $(\mu, k)$.