AN OVERVIEW OF TIME AND FREQUENCY LIMITING

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INTRODUCTION

This note aims to motivate and sketch informally some of the work on time and frequency limiting that I have seen at close quarters. The account is entirely subjective, and is not meant to speak for the friends and collaborators - notably H.O. Pollak, D. Slepian, and B.F. Logan - from whom I learned much of this material, and who could explain it far better. I also apologize to the many other contributors for inadequate mention of their work. What follows is impressionistic and incomplete, intending, as does any brief survey, only to show the interest and charm of the area, with the hope of enticing the reader to return at greater leisure.

A PARADOX

We can begin with the Fourier decomposition. A square-integrable function f(t) - which we term a function of finite energy - has a square-integrable Fourier transform $F(\omega)$, and

$$f(t) = \frac{1}{\sqrt{2\pi}} \int F(\omega) e^{i\omega t} \, d\omega. \qquad (1$$

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(Unless otherwise stated, all integrals will be over $(-\infty,\infty)$.) Viewing the integral of (1) as a decomposition into a sum of frequency components, and believing that physical devices - vocal chords, membranes, oscillators - all have upper limits on the rate at which they can vibrate, we can feel confident that in modeling the outputs of such devices the representation (1) may without loss of generality be corresponding truncated to

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} F(\omega) e^{i\omega t}\, d\omega, \tag{2}$$

for some $\Omega < \infty$, or, equivalently, that

$$F(\omega) \equiv 0, \quad |\omega| > \Omega.$$

We will call square-integrable functions of the form (2) frequency-limited - or, more specifically, band-limited - and denote that collection by $\underline{B}(\Omega)$. Of course, with the same confidence we can believe that physical devices have zero response when not activated, so that their outputs may with equal justice be assumed to have only finite duration

$$f(t) \equiv 0, \ |t| > T , \tag{3}$$

for some $T < \infty$. We call square-integrable functions of the form (3) time-limited, and denote that collection by $\underline{D}(T)$.

The trouble arises because, on extending the variable t in (2) to the complex (t+iu)-plane, we see that a band-limited function is analytic - indeed, entire - hence cannot vanish on any interval without vanishing identically. Thus f(t) cannot be simultaneously band-limited and time-limited. Another implication of analyticity is extrapolatability: a band-limited f(t) can be extrapolated everywhere from knowledge of it on an arbitrarily small interval (say by means of a power series expansion) so that, for example, any second's worth of speech would determine the entire utterance. Clearly, the assumption of band-limitedness seems to embroil us in paradox. And yet, in the practical design of communications systems, that assumption, and some of its theoretical consequences such as sampling, are exploited in an essential way without generating contradictions. Moreover, according to engineering experience, not only do there exist signals which are, for all purposes, simultaneously time and band-limited to $|t| < T$ and $|\omega| < \Omega$, respectively, but there are approximately $2\Omega T/\pi$ independent ones. Thus the problem of reconciling the models (2) and (3) presents an interesting challenge [31, 33].

CONCENTRATION AND THE UNCERTAINTY PRINCIPLE

If a band-limited f(t) cannot be supported on a finite interval $|t| < T$, we can nevertheless ask how well it can be concentrated (in energy) on the interval, i.e., how large we can make the ratio

$$\alpha_f^2 = \int_{-T}^{T} |f(t)|^2 dt / \int |f(t)|^2 dt,$$

for $f \in \underline{B}(\Omega)$. By standard analysis we can conclude that this ratio attains its supremum for a function $\phi_0(t) \in \underline{B}(\Omega)$, that ϕ_0 is the solution of the eigenvalue equation

$$\frac{1}{\pi} \int_{-T}^{T} \frac{\sin\Omega(t-s)}{t-s} \phi_0(s)\, ds = \lambda_0 \phi_0(t), \qquad (4)$$

corresponding to the largest eigenvalue λ_0, and that λ_0 is the desired $\sup(\alpha_f^2)$; a change of variable shows $\sup(\alpha_f^2)$; λ_0 to depend only on the product ΩT.

We can profitably interpret this result geometrically, by viewing the space L^2 of square-integrable functions as a Hilbert space, in which

$$(f,g) = \int f(t)\overline{g(t)} dt \ ,$$

$$||f||^2 = \int |f(t)|^2 dt \ ;$$

the Fourier transform is then a unitary operation. We observe that $\underline{D}(T)$ and $\underline{B}(\Omega)$ are closed linear subspaces, and that the (orthogonal) projections of L^2 onto these subspaces are given by the restrictions

$$D_T f = \begin{cases} f(t), & |t| < T \\ 0, & |t| > T, \end{cases}$$

$$B_\Omega f = \frac{1}{\pi} \int \frac{\sin \Omega(t-s)}{t-s} f(s)\, ds ,$$

respectively. The expression for B_Ω defines that function whose Fourier transform agrees with $F(\omega)$ in $|\omega| < \Omega$ and vanishes elsewhere; thus the operation B_Ω is completely analogous to D_T, and we refer to it as <u>frequency-limiting</u>. In terms of these projections, the concentration of interest becomes $\alpha_f^2 = ||D_T f||^2/||f||^2$. Visualizing the projection as in Euclidean space, α_f is the cosine of the angle which f, a vector in $\underline{B}(\Omega)$, forms with the subspace $\underline{D}(T)$; maximizing α_f then corresponds to finding the least angle between the two subspaces. Equation (4) asserts that if this angle is attained at $\phi_0 \in \underline{B}(\Omega)$ then

$$B_\Omega D_T \phi_0 = \lambda_0 \phi_0 , \tag{5}$$

i.e., that projecting ϕ_0 onto $\underline{D}(T)$ and then back onto $\underline{B}(\Omega)$ yields a vector aligned with ϕ_0. A simple picture in three dimensions will illustrate the geometric reasonableness of this.

The largest eigenvalue of (4) measures the least angle between $\underline{B}(\Omega)$ and $\underline{D}(T)$, and its associated eigenfunction ϕ_0 is the band-limited function best concentrated in $|t| < T$. As such, it and approximations to it have found use in filter design and spectral estimation [10,11,19]. It also plays a role in the study of uncertainty.

The uncertainty principle asserts that a function and its Fourier transform cannot both be concentrated on small intervals: narrowing one necessarily produces broadening of the other. The classical formulation of this considers $|f(t)|^2$ and $|F(\omega)|^2$ normalized as probability distributions,

$$\int |f(t)|^2 dt = \int |F(\omega)|^2 d\omega = 1,$$

and, gauging the spread of each by its variance,

$$\sigma_f^2 = \int t^2 |f(t)|^2 dt$$

$$\sigma_F^2 = \int \omega^2 |F(\omega)|^2 d\omega,$$

establishes that

$$\sigma_f \sigma_F \geq 1/2.$$

This result is fundamental in modern physics, where it is interpreted to mean that certain pairs of physical quantities cannot simultaneously be accurately determined. Interesting local versions have recently been proved [25].

With the thought that variance may be a somewhat imprecise measure of dispersion, we can examine uncertainty in terms of energy concentration over specified intervals. Accordingly, with T and Ω fixed, and any $f \in L^2$, let

$$\alpha_f^2 = \int_{-T}^{T} |f(t)|^2 dt / \int |f(t)|^2 \, dt,$$

$$\beta_f^2 = \int_{-\Omega}^{\Omega} |F(\omega)|^2 d\omega / \int |F(\omega)|^2 d\omega.$$

The set Σ of points (α_f, β_f) in the unit square, generated as f ranges through L^2, describes the possible time and frequency concentrations that are simultaneously attainable. Again, recognizing that $\alpha_f = ||D_T f||/||f||$ and $\beta_f = ||B_\Omega f||/||f||$, we see that these ratios are the cosines of the angles which f forms with $\underline{D}(T)$ and $\underline{B}(\Omega)$, respectively. It is then a short step to conclude that these angles cannot sum to less than the angle between the subspaces themselves, so that Σ is delimited by the curve $\cos^{-1} \alpha_f + \cos^{-1} \beta_f = \cos^{-1} \lambda_0$ (ellipse inclined at 45°), traced out by linear combinations of ϕ_0 and $D_T \phi_0$. The uncertainty principle reflects itself here in the fact that Σ is bounded away from (1,1) [12].

DOUBLE ORTHOGONALITY OF $\{\phi_k\}$

The problem of maximum concentration has led to the equation (4), which in turn generates other eigenfunctions

$\{\phi_k\}$, $k \geq 0$, and corresponding eigenvalues $\lambda_k = \lambda_k(\Omega T)$. The fundamental properties of an orthogonal projection P are

$$P^2 = P,$$
$$(Pf,g) = (f,Pg), \qquad (6)$$

for any f,g in the Hilbert space. Since $\phi_k \in \underline{B}(\Omega)$, we can rewrite the operator of (4) and (5) as $B_\Omega D_T B_\Omega$, which shows it to be positive and self-adjoint; it is also compact since the kernel of (4) is square-integrable. It follows that $\{\lambda_k\}$ are positive, approaching 0 as $k \to \infty$, and that $\{\phi_k\}$ are mutually orthogonal functions which, when normalized, form a basis for $\underline{B}(\Omega)$. Moreover, invoking (6),

$$(D_T\phi_i, D_T\phi_j) = (D_T B_\Omega \phi_i, D_T B_\Omega \phi_j) = (B_\Omega D_T^2 B_\Omega \phi_i, \phi_j) =$$

$$(B_\Omega D_T B_\Omega \phi_i, \phi_j) = \lambda_i \delta_{ij}.$$

Thus the eigenfunctions enjoy a remarkable property: not only are they mutually orthogonal over the entire t-axis, but their restrictions to $|t| \leq T$ are also mutually orthogonal and, when renormalized to $\{\lambda_k^{-1/2} D_T\phi_k\}$, form a basis for $\underline{D}(T)$. This double orthogonality makes $\{\phi_k\}$ an ideal basis for considering the many problems in which information about a square-integrable band-limited function is given on an interval. We illustrate with the question of extrapolation, already mentioned.

Suppose that $g(t)$ is given for $|t| \leq T$. We ask whether $g(t)$ is the restriction to $|t| \leq T$ of some $f \in \underline{B}(\Omega)$ and, if so, what is f? On expanding the given $g(t)$ in the basis $\{\lambda_k^{-1/2} D_T\phi_k\}$, we obtain

$$D_T g = \sum_{k=0}^{\infty} a_k \lambda_k^{-1/2} D_T \phi_k \ , \qquad (7)$$

where the coefficients a_k are given explicitly by

$$a_k = \int_{-T}^{T} g(t)\phi_k(t)\lambda_k^{-1/2} dt \ ;$$

being components of D_T g in an orthonormal basis, they satisfy

$$\Sigma\ |a_k|^2 = \int_{-T}^{T} |g(t)|^2 dt < \infty. \tag{8}$$

The right-hand side of (7), however, is $D_T(\Sigma\ a_k \lambda_k^{-1/2} \phi_k)$. This shows that the desired f must be

$$f(t) = \sum_{k=0}^{\infty} a_k \lambda_k^{-1/2} \phi_k$$

which defines an element of $\underline{B}(\Omega)$ if and only if

$$||f||^2 = \Sigma \frac{|a_k|^2}{\lambda_k} < \infty. \tag{9}$$

Since $\lambda_k \to 0$, the extrapolation problem is inherently unstable, or ill-posed in the sense of Hadamard: an insignificant change in $\{a_k\}$ can radically alter the convergence of (9). Of course, qualitatively, this is in no way surprising, since the given data must be analytic to be the restriction of a band-limited function, and analyticity is easily destroyed by arbitrarily small changes. But more specifically, we can see by restricting to finite sums in (7) and (9) that the instability here arises because small improvements in approximating g(t) in $|t| < T$, when contributed by a basis element ϕ_k for which the concentration λ_k is small, can entail arbitrarily large swelling of the size of f(t) away from the interval of observation $|t| \leq T$. To eliminate the instability we must therefore somehow reduce the influence of these components in (7). One natural way of doing this is to reformulate the extrapolation problem, by asking for the best approximation $D_T f$ to $D_T g$ among functions $f \in \underline{B}(\Omega)$ for which the norm has a fixed bound A, reserving the freedom to vary A. It is easy to see that f here is given by

$$f = \sum_{k=0}^{\infty} \frac{a_k}{\lambda_k^{1/2}\ (1+\frac{\mu}{\lambda_k})} \phi_k.$$

with μ chosen so that

$$\sum_{k=0}^{\infty} \frac{a_k^2}{\lambda_k (1+\frac{\mu}{\lambda_k})^2} = A^2.$$

A cruder approach might be simply to truncate the expansion (7) of $D_T g$ to those $\{\phi_k\}$ for which λ_k is not too small. To obviate the need for this, R.W. Gerchberg and A. Papoulis [5,5a] suggested the iteration

$$f_1 = B_\Omega D_T g$$

$$f_{n+1} = f_n - BDf_n + BDg, \quad n > 1.$$

By expanding in terms of $\{\phi_k\}$ it is again easy to see that this procedure generates

$$f = \Sigma\, a_k \lambda_k^{-1/2}\, \phi_k$$

by means of the series

$$\lambda_k^{-1/2} = \frac{\lambda^{1/2}}{1-(1-\lambda_k)} = \lambda_k^{1/2}[1+(1-\lambda_k) + (1-\lambda_k)^2 + \dots],$$

so that

$$f_n = \sum_{k=0}^{\infty} a_k \lambda_k^{-1/2}\, [1-(1-\lambda_k)^n]\, \phi_k,$$

a formula in which the weights $[1-(1-\lambda_k)^n]$ attached to the coefficients have, at least theoretically, the desired stabilizing characteristic of reducing the influence of small $\{\lambda_k\}$. Convergence here can be very slow, however [3,4].

WELL-CONCENTRATED FUNCTIONS AND THEIR APPROXIMATE DIMENSION

We recall that the largest eigenvalue $\lambda_0 = \lambda_0(\Omega T)$ of (4) represents the maximum concentration

$$||D_T f||^2/||f||^2$$

attainable by a bandlimited function $f \in \underline{B}(\Omega)$. However, the succeeding eigenvalues also have an interesting interpretation: by the minimax characterization [27, p.237], $\lambda_{k-1} = \lambda_{k-1}(\Omega T)$ is the maximum k^{th} largest concentration among k mutually orthogonal bandlimited functions. To pursue the well-concentrated functions we therefore focus on the behavior of eigenvalues; as these depend only on ΩT, let us henceforth normalize so that $\Omega = \pi$. Since a projection never increases the norm, we have $0 < \lambda_k < 1$, the strict inequalities a consequence of the analyticity of $f \in \underline{B}(\Omega)$. By applying to (4) the formulas for the trace and for the Hilbert-Schmidt norm, we find

$$\sum_{k=0}^{\infty} \lambda_k(\pi T) = \frac{1}{\pi}\int_{-T}^{T} \pi \, dt = 2T, \tag{10}$$

$$\sum_{k=0}^{\infty} \lambda_k^2(\pi T) = \int_{-T}^{T}\int_{-T}^{T} \frac{1}{\pi^2} \frac{\sin^2\pi(t-s)}{(t-s)^2} \, ds \, dt > 2T - c \log T, \tag{11}$$

for some c independent of T, so that

$$\sum_{k=0}^{\infty} \lambda_k(1-\lambda_k) < c \log T. \tag{12}$$

With $\delta > 0$ fixed let us now consider those $\{\lambda_k(\pi T)\}$ for which $\delta < \lambda_k(\pi T) < 1 - \delta$. Each of these contributes an amount no smaller than $\delta(1-\delta)$ to the left-hand side of (12), so that the number of such contributions grows no faster than $c_1 \log T$. Thus the eigenvalues are near 1, then near 0, the transition occurring over a relatively narrow range of values of k. Combining this with (10) and (11) shows that the number of large eigenvalues is $2T - c_2 \log T$; the corresponding eigenfunctions then span a subspace of $\underline{B}(\Omega)$ of that dimension, in which every function is well-concentrated on $|t| < T$. As for the intermediate eigenvalues, we can show by a separate argument that, more precisely, $\lambda_{[2T]}(\pi T)$ is bounded away from 0 and 1, independently of T [14], and indeed - by suitable choice of the function h(t) figuring in [14] - that

$$\lambda_{[2T]}(\pi T) \simeq 1/2.$$

(Here [x] denotes the integer part of x.)

On viewing $||f||^2$ as the energy of the function f(t), we can interpret the preceding results so as to suggest a possible resolution of the paradox of simultaneous time and bandlimitedness of physical signals. For suppose that our reading of energy is accurate only to an order of magnitude ε, so that we cannot distinguish between 0 and a function h(t) for which $||h||^2$ is of the order of ε. This suggests, firstly, that a bandlimited function f(t) whose energy outside an interval $|t| \leq T$ lies below the detection threshold will appear time-limited. Moreover, any approximation whose accuracy is commensurate with this unconcentrated energy will likewise be perceived as exact. Specifically, for $f \in \underline{B}(\pi)$, if

$$f = \sum_{k=0}^{\infty} a_k \phi_k,$$

let

$$g = \sum_{k \leq [2T]-1} a_k \phi_k .$$

Then

$$||f-g||^2 = \sum_{k \geq [2T]} |a_k|^2 \leq \frac{1}{1-\lambda_{[2T]}} \sum_{k \geq 0} |a_k|^2 (1-\lambda_k)$$

$$\leq c \int_{|t|>T} |f(t)|^2 dt,$$

with c a fixed constant which, in view of the behavior of $\lambda_{[2T]}(\pi T)$, can be taken as 2. It follows that, whatever the precision with which energy can be measured, those bandlimited functions which are perceived as time-limited (because the unconcentrated energy lies below the detection threshold) are likewise, to the same order of measurement accuracy, seen to lie in the [2T]-dimensional subspace spanned by $\phi_0, \dots, \phi_{[2T]-1}$. It is this notion of approximate dimensionality - much better justified and explained in [31]-which expresses the engineering intuition mentioned at the outset.

It is interesting that there are simpler-generated [2T]-dimensional subspaces which approximate $B(\Omega)$ as well as does that spanned by the eigenfunctions. An example is the subspace spanned by the translates $\sin \pi(t-y_k)/(t-y_k)$, with

$\{y_k\}$ the zeros of $\phi_0(t)$ in $|t| < T$ [21]. This is a delicate matter, however, for the translates of sin $\pi t/t$ centered at the integers in $|t| < T$ produce a far poorer approximation [13].

APPROXIMATE PROJECTIONS; EIGENVALUE DISTRIBUTION

A self-adjoint operator whose eigenvalues are 1 and 0 is necessarily a projection. We have seen that this property is nearly true of $D_T B_\Omega D_T$, which we can therefore picture as being close to a projection onto the subspace of functions in $\underline{B}(\Omega)$ well-concentrated on $|t| < T$, having dimension $2T\Omega/\pi - O(\log T)$. This point of view helps to clarify results on the asymptotic distribution, as $T \to \infty$, of eigenvalues of the integral equation

$$\frac{1}{\sqrt{2\pi}} \int_{-T}^{T} k(x-y)f(y)dy = \mu f(x), \quad |x| < T ,$$

when k is integrable. For on subdividing the ω-axis into intervals I_j over which $K(\omega)$, the Fourier transform of $k(x)$, is nearly a constant, γ_j, we can approximate this integral operator by

$$\Sigma \, \gamma_j D_T B_{I_j} D_T .$$

This is a linear combination of nearly orthogonal approximate projections, so that the eigenvalues behave like γ_j with multiplicity approximately T meas $(I_j)/\pi$. Letting $T \to \infty$ shows that, to first order, these eigenvalues have the same distribution as do the values of $K(\omega)$ sampled at the integer multiples of π/T. This result was proved for real-valued $K(\omega)$ in [9], but the present argument applies also to complex-valued $K(\omega)$ [16,17].

A much more refined question concerns the behavior of the $\{\lambda_i\}$ of (4) in the transitional region $0 < \lambda < 1$. The conjecture of [30], that $\lambda[2T+(b/\pi)\log T] \to 1/(1+\exp(\pi b))$, was established in [18], and far-reaching generalizations are now known [1,36].

EXTENSIONS

All but the very last of the results described depend only on simple features of Hilbert space geometry, and so apply without change when the intervals $|t| < T$ and $|\omega| < \Omega$, to which time and frequency are limited, are replaced by arbitrary compact sets, possibly in higher dimension [28, 29, 15, 6]. The structure also persists in discrete versions, as for periodic functions, where Fourier coefficients play the role of the Fourier transform [32]. Concentration and approximation problems for band-limited functions can also be posed in L^p spaces, $p \neq 2$, where - with orthogonal decomposition less ready to hand - they yield a rich variety of difficult questions. Extensive and deep results in this area have been discovered by B.F. Logan, and will appear in a forthcoming book [20]. Approximate dimension for bounded bandlimited functions has been elegantly formulated and determined in [22].

THE EIGENFUNCTIONS

In view of their many potential applications, it is valuable to find the eigenfunctions explicitly. The crucial observation here is that the integral of (4) commutes with a certain second-order differential operator, whence it follows that $\{\phi_k\}$ are the prolate spheroidal wave eigenfunctions of the latter [24]. A great deal of information, theoretical and computational, about local and asymptotic behavior flows from this identification; these results were derived with dazzling virtuosity in [30]. A program for calculating the $\{\phi_k\}$ is also available [23,35]. In discrete versions, where band-limited sequences replace band-limited functions, a tridiagonal matrix plays the role of the commuting differential operator, and permits rapid and accurate calculation of the analogous successively best-concentrated eigensequences.

Because of the light which the commutativity sheds on the eigenfunctions, it is interesting to inquire when it can take place. The question has been posed in generality by F.A. Grunbaum [7,8] who, having discovered many examples in a variety of analogous contexts, has asked for an explanation that would account for them all. This problem reaches far, and is still open.

SPECTRAL ESTIMATION

Spectral estimation seeks to produce a plausible decomposition into frequency components of a given function f(t), generally obtained empirically over a finite interval: $f \in D(T)$ for some T. As the purpose is to elucidate the process which generated the data, interest often focuses on the extent to which this frequency content is concentrated on some subset of frequencies, not known in advance; in this way, the problem is connected with our subject. The best approximation to a function by a component having frequencies in an interval I is given by applying the projection B_I to the function, but as the sinx/x convolution kernel of B_I has considerable energy in its tail, this frequency-limiting operation is computationally inaccurate, especially for small I, and if used on $D_T f$ is too sensitive to the unobserved values of f in $|t| > T$. Convergence here can be improved by substituting for that kernel the best-concentrated eigenfunction ϕ_0 of $B_I D_U B_I$, with a suitable $U < T$, for since the energy of ϕ_0 in $|s| > U$ is small, the error contributed to the convolution of f and ϕ_0 by the time-limiting of f can be reduced in the range $|t| < T-U$; the choice of U effects a trade among the sizes of this range, of the error, and of I. The existence of an easily calculated accurate approximation to ϕ_0 makes this approach practical and useful [10, 11]. Recently, D.J. Thomson has suggested that the data could be utilized more fully, and in a way which has many desirable statistical characteristics, if the component of f(t) with frequencies in I were approximated by projecting $D_T f$ onto the subspace spanned by the first few most-concentrated eigenfunctions of $B_I D_T B_I$. The total decomposition of f is then generated in this way by sweeping I over the frequencies. Here, when I_1 and I_2 are disjoint, the highly concentrated eigenfunction from $\underline{B}(I_2)$ are nearly orthogonal over $|t| < T$ to those from $\underline{B}(I_2)$, so that this extraction of frequency components proceeds by nearly orthogonal increments; when I_1 and I_2 overlap, an average is performed of the components determined in the common region. For the more usual case when the data consists of equally spaced samples, rather than of a continuous record, the eigenfunctions are replaced by eigensequences, which are readily calculated. This method has proved to be outstandingly successful [19, 34].

SAMPLING AND INTERPOLATION

The most important practical consequence of band-limitedness is that function of $\underline{B}(\Omega)$ are equivalent, in a stable way, to their values at the points $\{k\Omega/\pi\}$. Specifically, normalizing $\Omega = \pi$, if we write $F(\omega)$, $|\omega| < \pi$, as a Fourier series, we obtain

$$F(\omega) = \sum_{-\infty}^{\infty} a_n e^{-in\omega}/\sqrt{2\pi}, \tag{13}$$

where

$$a_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} F(\omega)e^{ik\omega}d\omega = f(k) , \tag{14}$$

$$\Sigma |f(k)|^2 = \Sigma |a_k|^2 = \int_{-\pi}^{\pi} |F(\omega)|^2 d\omega = \int |f(t)|^2 dt . \tag{15}$$

On applying the Fourier transform to (13) we find, by (14),

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} F(\omega)e^{i\omega t}d\omega = \Sigma_{-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)} , \tag{16}$$

the series converging in L^2, and, by Schwarz's inequality, also uniformly. The formula (16) is thus an expansion in the orthonormal basis of functions $\{\sin \pi(t-n)/\pi(t-n)\}$, which simultaneously form an interpolating family at the sampling points $\{t = n\}$. The orthonormality ensures stability, for an imprecision ε_n in reading the sample value $f(n)$ will lead to a reconstruction whose error,

$$e(t) = \Sigma\, \varepsilon_n \sin \pi(t-n)/\pi(t-n),$$

will, by (15), have norm equal to the measurement error $\Sigma|\varepsilon_n|^2$. The interpolating property of the basis functions ensures that arbitrary square-summable values can likewise be stably realized as the sample values of a simply constructed band-limited function of finite energy. Thus (16) provides a stable way of passing between continuous band-limited and discrete information - the foundation on which modern telecommunication systems are built. The rate of sampling, Ω/π, at which this is done is often called the Nyquist rate.

In view of the usefulness of (16), it is natural to ask whether functions of $\underline{B}(\Omega)$ can be recovered from their values at other, perhaps sparser, sets of points; this would make discretization of band-limited signals more efficient. Likewise, we can try to find denser sets of points at which arbitrary square-summable values $\{a_k\}$ can be interpolated by some $f \in \underline{B}(\Omega)$; this would make band-limited transmission of discrete information more efficient.

To examine these questions, we consider the following possible features of a given set of real points Λ.

A. Λ is a set of uniqueness for $\underline{B}(\Omega)$ if each $f \in \underline{B}(\Omega)$ must vanish identically if it vanishes at all the points of Λ.

B. Λ is a set of stable sampling for $\underline{B}(\Omega)$ if the points of Λ are uniformly separated (i.e., there is $\delta > 0$ such that for distinct $\tau_i, \tau_j \in \Lambda, |\tau_i - \tau_j| > \delta$) and if there is a constant K such that, for each $f \in \underline{B}(\Omega)$,

$$\int_{-\infty}^{\infty} |f(t)|^2 dt < K \sum_{\tau \in \Lambda} |f(\tau)|^2 .$$

C. $\Lambda = \{\tau_k\}$ is a set of interpolation for $\underline{B}(\Omega)$ if, corresponding to each square-summable sequence $\{a_k\}$, there exists $f \in \underline{B}(\Omega)$ with $f(\tau_k) = a_k$.

If Λ is a set of uniqueness for $\underline{B}(\Omega)$, then two functions of $\underline{B}(\Omega)$ which agree at the points of Λ must coincide; thus the sample values $\{f(\tau), \tau \in \Lambda\}$, determine f everywhere. Sets of uniqueness can be very sparse. For example, invoking the analyticity of f, we see that any collection of points with a point of accumulation will qualify. To exhibit more uniformly distributed Λ, we can take integer multiples of $\alpha > \Omega/\pi$, only on a half-line. There are many kinds of examples, and their variety seemed chaotic until, in what is undoubtedly the deepest result in a vast area, A. Beurling and P. Malliavin discovered a density whose size controls the feature of uniqueness [2,26].

Although $f \in \underline{B}(\Omega)$ is determinable from its values on a set of uniqueness, this fact does not constitute a sampling reconstruction adequate in practice. For, just as in the problem of extrapolating from a finite interval, an arbitrarily small error in measuring the sample values can well lead to arbitrarily large errors in the reconstructed signal, or indeed can make the extrapolation impossible. In order to ensure stability of reconstruction, we require that Λ be a set of stable sampling for $\underline{B}(\Omega)$. In sharp contrast to sets of uniqueness, the points of a set of stable sampling must be regularly distributed and, on all sufficiently large intervals, must have at least the Nyquist rate [14]. For an intuitive explanation, let us take an interval I of length r, let n(r) be the number of $\{\tau_k\}$ in I, and let us consider the set S of function of $\underline{B}(\Omega)$ well-concentrated on I, so that the energy of $f \in S$ outside I is small. If we can presume that the samples of $f \in S$ at the points $\{\tau_k\}$ outside I are also small, then, by virtue of stability, replacing these sample values by zero should not greatly affect our reconstruction of f(t). We conclude that a function of S is substantially determined by n(r) measurements, so that the collection S is, in a sense, no more than n(r)-dimensional. But we have shown the approximate dimension of S to be at least $r\,\Omega/\pi - c \log r$. Thus $n(r) > r\,\Omega/\pi - c \log r$ from which $\lim_{r\to\infty} n(r)/r > \Omega/\pi$, and our assertion follows.

For sets of interpolation, similar considerations apply, and show that the maximum density on all large intervals cannot exceed Ω/π. Thus the Nyquist rate for sampling and interpolation of band-limited functions cannot be improved [14]. Again, since the argument is based only on the first-order behavior of the eigenvalues of (4), it can be carried out without change for more complicated frequency sets than a single interval, and in more dimensions [15].

CONCLUSION

Band-limited functions possess many properties that stem from their analyticity. However, as analyticity is fragile, not all of these persist under small perturbation. If we require that our conclusions remain stable when functions are determinable only with given precision, we are led to problems in which the time-and-frequency-limiting operator enters naturally. The eigenvalues of this operator show that, to first order, it resembles a projection, and the stable properties of band-limited functions typically inherit from it a simple and orderly behavior, useful in theory and application.

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