ADDENDUM TO LECTURE 9

...ting property

$$x(t-t_0) \xrightarrow{FT} e^{-j\omega t_0} X(j\omega)$$

$$\int_{-\infty}^{+\infty} x(t-t_0)\, e^{-j\omega t}\, dt$$ just make a change of variables $t - t_0 = t'$

$$= \int_{-\infty}^{+\infty} x(t')\, e^{-j\omega(t'+t_0)}\, dt'$$

$$= \int_{-\infty}^{+\infty} x(t')\, e^{-j\omega t'}\, dt' \; e^{-j\omega t_0}$$

$$= X(j\omega)\, e^{-j\omega t_0}$$

Analogously for a shift in freq.

$$X(j(\omega-\omega_0)) \longleftrightarrow x(t)\, e^{j\omega_0 t}$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j(\omega-\omega_0))\, e^{j\omega t}\, d\omega$$ call $\omega - \omega_0 = \omega'$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega')\, e^{j(\omega'+\omega_0)t}\, d\omega$$

$$= x(t)\, e^{j\omega_0 t}$$

This turns out to be useful to compute:

$$FT[e^{j\omega_0 t}]$$

$$= \int_{-\infty}^{+\infty} e^{j\omega_0 t} e^{-j\omega t}\, dt = \int_{-\infty}^{+\infty} 1\, e^{-j(\omega-\omega_0)t}\, dt$$

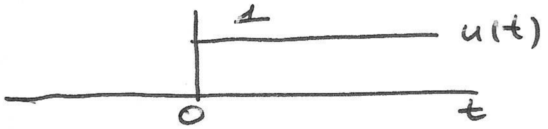
$$= 2\pi\,\delta(\omega-\omega_0)$$

recall $FT(1) = 2\pi\delta(\omega)$
and property of shift
in freq

On the other hand

$$\delta(t-t_0) \longleftrightarrow e^{-j\omega_0 t_0}$$

Another useful property of δ-function
It is the derivative of step-function



$$u(t) \xleftrightarrow{FT} \pi\delta(\omega) + \frac{1}{j\omega}$$ (This is shown at the end of lecture notes 9)

Consider now

$\dfrac{du(t)}{dt}$ this is discontinuous at $t=0$
so the derivative is 0 for $t<0$
0 for $t>0$
how about at $t=0$?
It turns out that

$$\delta(t) = \frac{du}{dt}$$

In fact we write

$$\frac{du(t)}{dt} = FT^{-1}\left[FT\left(\frac{du}{dt}\right)\right]$$

and using the derivative property of FT we get

$$FT^{-1}\left[j\omega \; FT(u(t))\right] =$$

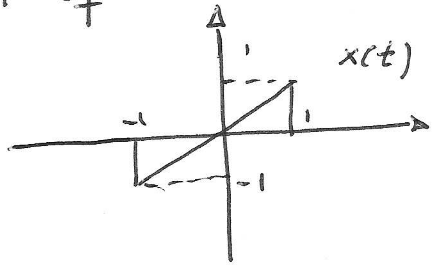
$$= FT^{-1}\left[j\omega\pi\delta(\omega) + 1\right] =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} 1\, e^{j\omega t} d\omega + \frac{1}{2\pi}\int_{-\infty}^{+\infty} j\omega\pi\, e^{j\omega t}\,\delta(\omega)\, d\omega$$

$$= \delta(t) \qquad + \qquad 0$$

because $FT(\delta)=1$ because $\int_{-\infty}^{+\infty} f(\omega)\,\delta(\omega)\,d\omega = f(0)$

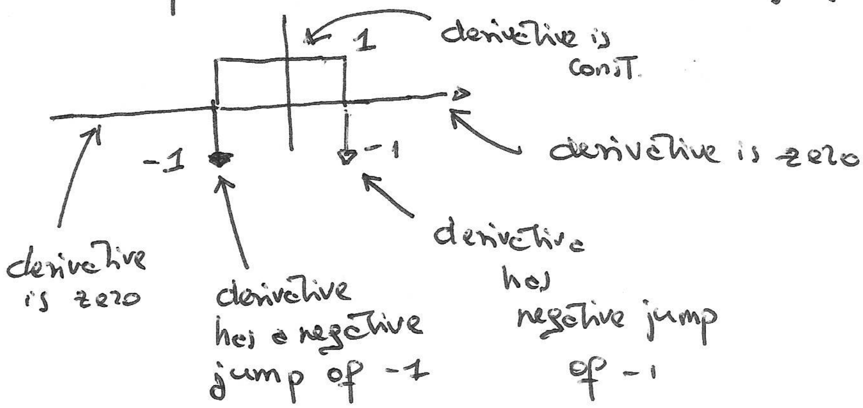
This property turns out to be useful:
For example if we want to compute
FT of



derivative of discontinuity correspond δ-function of amplitud equal to the "jump" of function

we may note that this can be written as:

$$\frac{dx}{dt} = -\delta(t+1) - \delta(t-1) + Rect(2,1)$$



Now recall the FT of rectangle of width 2 and height 1 we computed in previou lectures

$$FT\left(Rect(2,1)\right) = \frac{2\sin\omega}{\omega}$$

So we get

$$FT\left(\frac{dx}{dt}\right) = \frac{2\sin\omega}{\omega} \cancel{+}\ e^{-j\omega} - e^{+j\omega} = G(\omega) \qquad \text{note } G(0) = 0$$

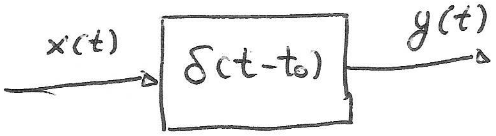
using shifting property

$$FT\left[x(t)\right] = \frac{1}{j\omega} G(j\omega) + \cancel{\pi G(0)\delta(\omega)}^{\,0} = \frac{2\sin\omega}{j\omega^2} - \frac{e^{-j\omega} - e^{+j\omega}}{j\omega}$$

integration property

$$= \frac{2\sin\omega}{j\omega^2} - \frac{2\cos\omega}{j\omega}$$

What does a system with impulse response $h(t) = \delta(t-t_0)$ do?

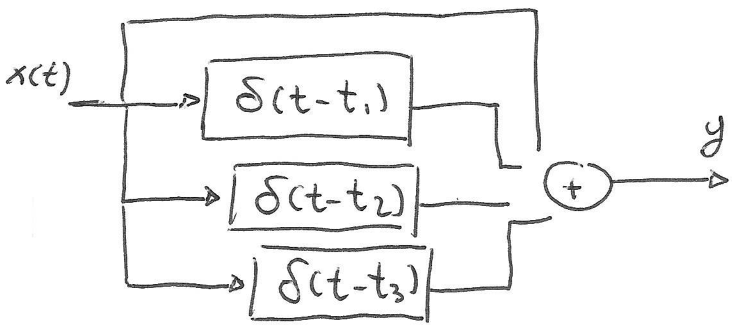


$$y(t) = FT^{-1}\left[X(j\omega)\, e^{-j\omega t_0} \right] = x(t-t_0)$$

Again time-shift property

multiplication by complex expo in freq. is a time shift

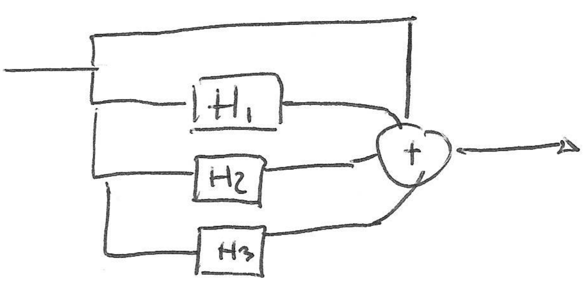
So, here is a cool device



$y(t) = x(t) + x(t-t_1) + x(t-t_2) + \cdots$

ECHO MACHINE!

What is the frequency response?



$|H_i| = 1$

$\angle H_i = -\omega t_0$

linear phase shift of slope $-t_0$.

Let

$$h(t) = \frac{\sin(at - 17a\pi^2)}{t - 17\pi^2} = \frac{\sin(a(t - 17\pi^2))}{t - 17\pi^2}$$

Compute FT(h(t)). First, we get from Table that:

$$FT\left(\frac{\sin at}{t}\right) = \pi \frac{\sin(at)}{\pi t} = \pi \ \text{rect}(\omega, a) = \begin{cases} \pi & |\omega| < a \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

So applying (1) & time-shift we get

$$FT(h(t)) = \pi \ \text{rect}(\omega, a)\, e^{-j\pi^2 17\omega} = \begin{cases} \pi e^{-17\pi^2\omega} & |\omega| < a \\ 0 & \text{othewise} \end{cases}$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Assume $\boxed{a \gg 1/\pi}$ $\boxed{a \ll \frac{2}{\pi}}$ for example $a = \frac{1.5}{\pi}$

What is output if input is $\cos\left(\frac{t}{\pi}\right)$?

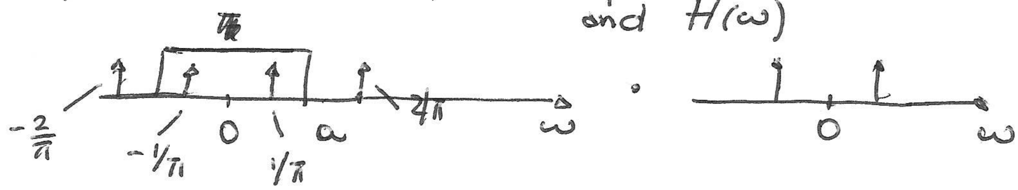
$$\boxed{y(t) = \pi \cos\left(\frac{t}{\pi} - \frac{17\pi^2}{\cancel{\pi}}\right)}$$

because $\cos(\omega_0 t)$ $\omega_0 = \frac{1}{\pi} < a$

What is output if input is $\cos\left(\frac{2t}{\pi}\right)$? because $\cos(\omega_0 t)$ $\omega_0 = \frac{2}{\pi} > a$

$$y(t) = 0$$

Graphically, we have the multiplication of $FT(\cos \omega_0 t) = \frac{1}{2}\delta(\omega - \omega_0) + \frac{1}{2}\delta(\omega + \omega_0)$ and $H(\omega)$



And then apply the phase shift.

Let:

$$y(t) = z(t)\cos(\omega_0 t) = z(t)\,\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

Compute $FT[y(t)]$

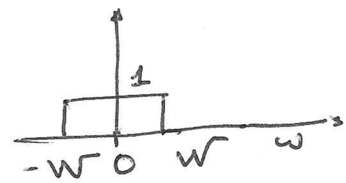
$$Y(j\omega) = \frac{Z(j(\omega+\omega_0)) + Z(j(\omega-\omega_0))}{2} \qquad (1)$$

shift property

If $z(t) = \dfrac{\sin Wt}{\pi t}$ what is $Y(j\omega)$?

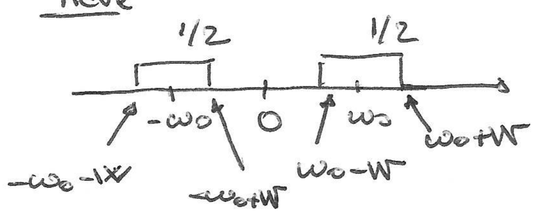
Recall

$$Z(j\omega) = \text{rect}[1, 2W] = \begin{cases} 1 & |\omega| < W \\ 0 & \text{otherwise} \end{cases}$$



$\Rightarrow$ { so we have

$Y(j\omega) =$



$$= \begin{cases} 1/2 & \omega \in [\omega_0+W, \omega_0-W] \text{ or } \omega \in [-\omega_0-W, -\omega_0+W] \\ 0 & \text{otherwise} \end{cases}$$

A general property: multiplication with cosine makes two copies of spectrum centered at ω_0 and $-\omega_0$ and scaled by 1/2

This is eq. (1)