UC San Diego J. Connelly

ECE 45 Average Power Review

Complex Power

When dealing with time-dependent voltage and currents, we have to consider a more general definition of power. We can calculate the *instantaneous power* at any point in time, but we often also care about the *average power over an interval of time*, which is independent of time but generally does depend on the interval.

$$P_{inst}(t) = v(t)i(t) \qquad P_{avg} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} P_{inst}(t)\,dt$$

When dealing with sinusoidal voltages and currents, we usually care about the average power in a *period*, i.e. $t_2 = T + t_1$, where T is the period of the sinusoid. In this case, the expression for average power simplifies to

$$P_{avg} = \frac{1}{2}\mathrm{Re}\{VI^*\}$$

where V and I are the phasor representations of $v(t)$ and $i(t)$ respectively.

Maximum Power

Suppose we want to maximize the average power delivered to a particular component in a circuit by adjusting the impedance of the component. In order to do this, we represent the circuit as a Thevenin equivalent, and we use *impedance matching* to find the values of current and voltage which maximize the power. That is we set $Z_L = Z_{th}^*$ where Z_{th} is the Thevenin impedance of the circuit.

Example 1

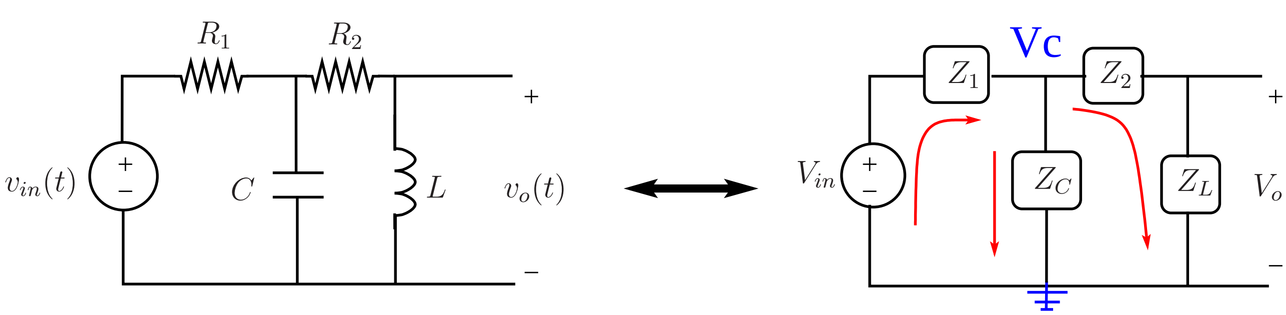
(a) *For any $\omega \geq 0$, find the output $v_o(t)$ when $v_{in}(t) = \cos(\omega t)$.*

(b) *How does the output behave as $\omega \to 0$ and $\omega \to \infty$*

(c) *When $v_{in}(t) = 2\cos^2(t)$, find $v_o(t)$.*

(d) *Suppose a **load** consisting of a resistor in series with a capacitor is placed across the output terminals. When $v_{in}(t) = \cos(2t)$, find the values of the resistor and the capacitor that maximize the average power deliver to the load.*

(e) *What is the maximum average power delivered to the load in this case?*



Please report any typos/errors to j2connelly@uscd.edu

Where $R_1 = 1\Omega$, $R_2 = 3\Omega$, $C = 1F$, $L = 1H$

Solutions

(a) By taking the phasor transform of the circuit with respect to ω and using a voltage divider, we can write the phasor of the output voltage as:

$$V_o = V_C \frac{Z_L}{Z_{R_2} + Z_L} \to V_C = V_o \frac{Z_{R_2} + Z_L}{Z_L} \tag{1}$$

By KCL at the node V_C:

$$\begin{aligned}\frac{V_{in} - V_C}{Z_{R_1}} &= \frac{V_C}{Z_C} + \frac{V_C}{Z_L + Z_{R_2}} \\ \to \frac{V_{in}}{Z_{R_1}} &= V_C \left(\frac{1}{Z_C} + \frac{1}{Z_L + Z_{R_2}} + \frac{1}{Z_{R_1}} \right) \\ \to V_C &= \frac{V_{in}}{Z_{R_1} \left(\frac{1}{Z_C} + \frac{1}{Z_L + Z_{R_2}} + \frac{1}{Z_{R_1}} \right)}.\end{aligned} \tag{2}$$

By setting equal 1 and 2:

$$V_o \frac{Z_{R_2} + Z_L}{Z_L} = \frac{V_{in}}{\frac{Z_{R_1}}{Z_C} + \frac{Z_{R_1}}{Z_L + Z_{R_2}} + 1}$$

V_{in} is the phasor of $v_{in}(t) = \cos(\omega t)$, so $V_{in} = 1$ and

$$\begin{aligned}V_o &= \frac{Z_L}{(Z_{R_2} + Z_L)(1 + \frac{Z_{R_1}}{Z_C}) + Z_{R_1}} = \frac{j\omega L}{(R_2 + j\omega L)(1 + j\omega R_1 C) + R_1} \\ &= \frac{j\omega L}{R_1 + R_2 - \omega^2 R_1 L C + j\omega(L + R_1 R_2 C)} = \frac{j\omega}{4 - \omega^2 + 4j\omega} = \frac{j\omega}{(j\omega + 2)^2}\end{aligned}$$

and so, solving for magnitude and phase of V_o yields:

$$|V_o| = \frac{|\omega|}{4 + \omega^2}, \quad \angle V_o = \frac{\pi}{2} - 2\tan^{-1}\left(\frac{\omega}{2}\right).$$

Transforming back to the time-domain gives us:

$$v_o(t) = \frac{|\omega|}{4 + \omega^2} \cos\left(\omega t + \frac{\pi}{2} - 2\tan^{-1}\left(\frac{\omega}{2}\right)\right).$$

(b) We have

$$\lim_{\omega \to 0} \frac{|\omega|}{4 + \omega^2} = 0$$

and

$$\lim_{\omega \to \infty} \frac{|\omega|}{4 + \omega^2} = 0$$

so the output is 0 at very high and very low frequencies This can be verified by looking at how the components in the circuit behave at high and low frequencies.

(c) Note that

$$2\cos^2(t) = 2\left(\frac{e^{jt}+e^{-jt}}{2}\right)^2 = \frac{2+e^{j2t}+e^{-j2t}}{2} = 1+\cos(2t)$$

Then $v_{in}(t) = v_{i,1}(t) + v_{i,2}(t)$, where $v_{i,1}(t) = 1$ and $v_{i,2}(t) = \cos(2t)$. Since $v_{i,1}(t)$ and $v_{i,2}(t)$ are different frequencies we need to use super position to find $v_o(t)$.

$v_{i,1}(t)$ is a sinusoid of frequency $\omega_1 = 0$, i.e. a DC signal, so plugging $\omega = 0$ into our expression for $v_o(t)$ yields

$$v_{o,1}(t) = 0$$

$v_{i,2}(t)$ is a sinusoid of frequency $\omega_2 = 2$, so plugging $\omega = 2$ into our expression for $v_o(t)$ yields

$$\frac{1}{4}\cos(2t)$$

Thus by superposition $v_o(t) = v_{o,1}(t) + v_{o,2}(t) = \frac{1}{4}\cos(2t)$.

(d) By the maximum power theorem, we need to find the Thevenin impedance of the circuit and match the load impedance. To find Z_{th}, set $V_i = 0$ and solve for the effective impedance at $\omega = 2$. We have

$$Z_{th} = (Z_1//Z_C + Z_2)//Z_L$$

$$Z_1//Z_C = \frac{R_1/j\omega C}{R_1 + j\omega C} = \frac{1}{2j+1}$$

$$Z_2 + Z_1//Z_C = \frac{6j+4}{2j+1}$$

$$Z_{th} = 1 + j\frac{3}{2}$$

The value of Z_{Load} that maximizes power to the load is $Z_{Load} = Z_{th}^* = 1 - j3/2$.
We also have $Z_{Load} = R_{Load} + \frac{1}{j\omega C_{Load}}$.

Thus $R_L = 1$ and $C_L = 1/3F$.

(e) By using a voltage divider in the Thevenin equivalent circuit, the voltage across the load is:

$$V_{Load} = V_{th}\frac{Z_{Load}}{Z_{Load}+Z_{th}} = V_{th}\frac{2-j3}{4}$$

and the current through the load is

$$I_{Load} = \frac{V_{th}}{Z_{Load}+Z_{th}} = \frac{V_{th}}{2}$$

Hence the average power is

$$P_{avg} = \frac{1}{2}\,\mathsf{Re}\left\{V_{Load}I_{Load}^*\right\} = \frac{1}{2}\,\mathsf{Re}\left\{\frac{|V_{th}|^2(2-j3)}{8}\right\} = \frac{|V_{th}|^2}{8}$$

To find V_{th}, we use our results from part c. that showed that when the input is $v_{in} = \cos(2t)$, the (open circuit) output is $v_o = \cos(2t)/4$, so the phasor of the open circuit voltage is. $V_{th} = 1/4$ Thus

$$P_{avg} = \frac{1}{128}$$

Example 2

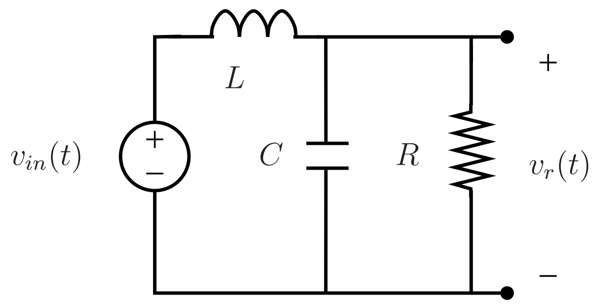
Find the voltage $v_r(t)$ in the circuit below, when

(a) $v_{in}(t) = 1/3$

(b) $v_{in}(t) = \sin(t)$

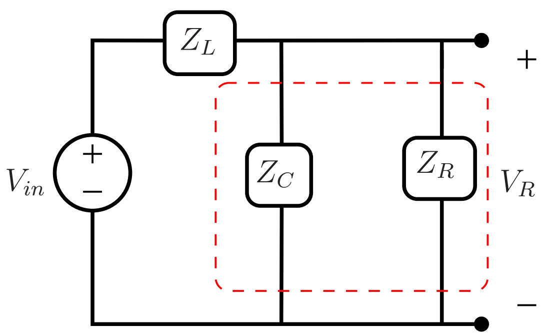
(c) $v_{in}(t) = 1 + 2\sin(t)$.

where $R = 1\Omega$, $C = 1/2F$, $L = 2H$.



Solutions

For $\omega \geq 0$, assume that $v_{in}(t) = A\cos(\omega t + \theta)$. Then by taking the phasor transformation of the circuit, we have



Voltage divider:

$$V_r = V_{in}\frac{Z_C//Z_R}{Z_C//Z_R + Z_L}.$$

where $Z_R = 2$, $Z_C = \frac{2}{j\omega}$, and $Z_L = 2j\omega$.

Since $V_{in}(t) = A\cos(\omega t + \theta)$, we have $V_{in} = Ae^{j\theta}$ and

$$V_R = \frac{Ae^{j\theta}}{1 + Z_L/Z_R + Z_L/Z_C} = \frac{Ae^{j\theta}}{1 + 2j\omega + (j\omega)^2} = \frac{Ae^{j\theta}}{(j\omega + 1)^2}$$

(a) In this case, we have $\omega = 0$ and $V_{in} = 1/3$, so

$$V_r = 1/3 \longrightarrow v_r(t) = 1/3.$$

(b) In this case, we have $\omega = 1$ and $\theta = -\pi/2$, so $V_{in} = e^{-j\pi/2}$ and

$$V_r = \frac{e^{-j\pi/2}}{\left(\sqrt{2}e^{j\pi 4}\right)^2} = -\frac{1}{2} \longrightarrow v_r(t) = -\frac{1}{2}\cos(t).$$

(c) Here we utilize the fact an RLC circuit is linear (i.e. super-position).

Let $v_{in}^{(1)}(t) = 1/3$ and $v_{in}^{(2)}(t) = \sin(t)$ and for each $k = 1, 2$, let $v_r^{(k)}(t)$ be the voltage across the resistor when $v_{in}^{(k)}(t)$ is the input voltage.

Then by parts (a) and (b), we have $v_r^{(1)}(t) = 1/3$ and $v_r^{(2)}(t) = -\frac{1}{2}\cos(t)$.

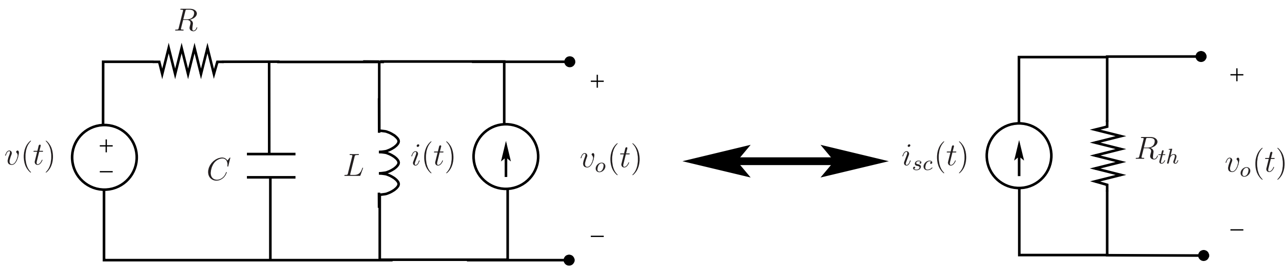
We have $v_{in}(t) = 3v_{in}^{(1)}(t) + 2v_{in}^{(2)}(t)$, so by linearity and time invariance, we have

$$v_r(t) = 3v_r^{(1)}(t) + 2v_r^{(2)}(t) = 1 - \cos(t).$$

Example 3

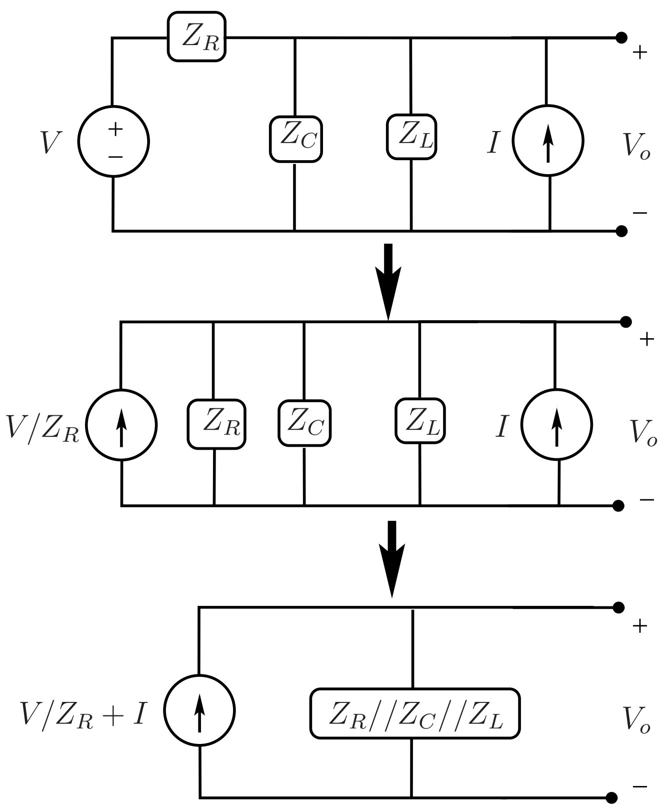
Recall the Norton Equivalent of an RLC circuit is a current source in parallel with a resistor and a capacitor or an inductor.

Find the value of C for which the Norton Equivalent is a current source in parallel with only a resistor (i.e. the Thevenin Impedance is purely real). What are $i_{sc}(t)$ and R_{th} in this case?



where $v(t) = \cos(4t + \pi/3)$, $i(t) = \sin(4t + 5\pi/6)$, $R = 1\Omega$, *and* $L = 1/4H$. $C =??$

Solutions



Since $v(t)$ and $i(t)$ are both sinusoidal with frequency 4, we can take the phasor transform with respect to $\omega = 4$.

Then by using a source transformation on the voltage source and Z_R, we have: $I_{sc} = I + V/Z_R$ and

$$\begin{aligned} Z_{eff} &= Z_R//Z_C//Z_L \\ &= \frac{1}{\frac{1}{Z_R} + \frac{1}{Z_C} + \frac{1}{Z_L}} = \frac{1}{1 + j4C - j}. \end{aligned}$$

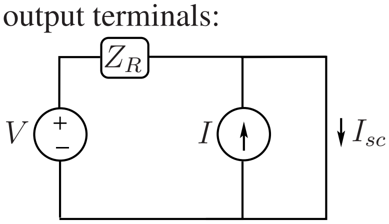
If Z_{eff} is real, then $C = 1/4F$.

Finally,

$$\begin{aligned} i_{sc}(t) &= i(t) + \frac{v(t)}{R} = 2\cos(4t + \pi/3) \\ R_{th} &= 1\Omega. \end{aligned}$$

Alternatively

We can solve for I_{sc} by shorting the output terminals:



We can solve for Z_{th} by setting $V = 0$ and $I = 0$ and solving for the effective impedance across the output terminals: