ECE 45 Discussion 1 Notes

Phasor Representation of Sinusoidal Functions

- Phasors are used to represent sinusoidal functions and allow for easier representation of linear (resistor, capacitor, and inductor) circuits with sinusoidal voltages and currents.
- Represent $A \cos(\omega t + \phi)$ as $A e^{j\phi}$.
- We do not include the frequency ω in the representation, but it is implicit.
- Differentiation and Integration:

$$f(t) = A \cos(\omega t + \phi) \longleftrightarrow A e^{j\phi} = F$$
$$\frac{df(t)}{dt} \longleftrightarrow j\omega F$$
$$\int_{-\infty}^{t} f(\tau) d\tau \longleftrightarrow \frac{1}{j\omega} F$$

Note: We can only use phasor representation when our function (input to circuit) is sinusoidal!

Impedance

When the inputs to our circuit are sinusoidal, we can represent resistors, capacitors, and inductors as generalized components called impedances. This allow for linear Voltage/Current relationships in the *phasor domain*.

$$v_R(t) = i_R(t) R \qquad \longleftrightarrow \quad V_R = I_R R$$
$$v_C(t) = \frac{1}{C} \int_{-\infty}^t i_C(\tau) d\tau \iff V_C = \frac{I_C}{j\omega C}$$
$$v_L(t) = L \frac{di_L(t)}{dt} \qquad \longleftrightarrow \quad V_L = j\omega L I_L$$

We can lump together the terms to end up with a general expression: V = I Z so

$$Z_R = R, \quad Z_C = \frac{1}{j\omega C}, \quad Z_L = j\omega L.$$

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Steady State Analysis

Because the impedance equation (V = IZ) has the same structure as Ohm's Law (v = iR), we can use circuit analysis techniques from DC circuit analysis such as:

- Parallel/Series Combinations
- KCL and KVL Analysis
- Source Transformations
- Voltage/Current Dividers
- Thevenin and Norton Equivalence

Example 1

Represent the following sinusoidal function as phasors as a single complex number in rectangular form:

$$2\cos(4\pi t + \pi/4) - 3\sin(4\pi t - \pi/3).$$

Solutions

Both terms have $\omega = 4\pi$ so we can write the function in phasor form:

$$2e^{j\pi/4} - 3e^{-j5\pi/6} = 2\left(\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right) - 3\left(-\frac{\sqrt{3}}{2} - \frac{j}{2}\right) = \frac{1}{2}\left(2\sqrt{2} + 3\sqrt{3} + j\left[2\sqrt{2} + 3\right]\right)$$

Example 2

Assume $\omega = 2\pi$ and represent the following phasor in sinusoidal form: $X = \frac{1}{1+j} e^{-j\pi/6}$.

Solutions

$$X = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}} e^{-j\pi/6} = \frac{1}{\sqrt{2}} \frac{e^{-j\pi/6}}{e^{j\pi/4}} = \frac{\sqrt{2}}{2} e^{-j5\pi/12} \iff x(t) = \frac{\sqrt{2}}{2} \cos(2\pi t - 5\pi/12)$$

Example 3

For what frequencies is the circuit component below purely resistive? (i.e. $Z_{eff} = X + j 0 = X$) What is the effective resistance at each frequency?



Solutions

$$Z_{eff}(\omega) = Z_C / / Z_{R_1} + Z_{R_2} + Z_L = \dots = \left(R_2 + \frac{R_1}{(R_1 C \omega)^2 + 1} \right) + j \left(\omega L - \frac{R_1^2 C \omega}{(R_1 C \omega)^2 + 1} \right)$$

The impedance is purely resistive if the imaginary portion of $Z_{eff}(\omega)$ equals 0:

$$0 = \omega L - \frac{R_1^2 C \omega}{(R_1 C \omega)^2 + 1} = \omega^3 R_1^2 L C^2 + \omega (L - R_1^2 C) = \omega \left(\frac{1}{2}\omega^2 - 1\right)$$

$$\therefore \ \omega_0 = 0, \ \omega_1 = \sqrt{2}, \ \omega_2 = -\sqrt{2}$$

and so the effective resistance at each frequency is:

$$Z_{eff}(0) = R_2 + R_1 = 3\sqrt{2}\Omega$$
$$Z_{eff}(\pm\sqrt{2}) = R_2 + \frac{R_1}{2R_1^2C^2 + 1} = 2\sqrt{2}\Omega$$

Example 4

In the circuit below, find $i_o(t)$ as a sine function.



Solutions

Since the current and voltage sources are both of the same frequency, we can represent the voltages and currents as phasors and the resistors, capacitor, and inductor as impedances:



By Ohm's Law, we have:

$$I_o = \frac{V_A}{Z_{R_1} + Z_L}, \quad I_2 = \frac{V - V_A}{Z_{R_2} + Z_C}$$

and by KCL we have: $I_{in} + I_2 = I_o$. By substituting in I_o and I_2 expressions, we have

$$I_{in} + \frac{V_{in} - V_A}{Z_{R_2} + Z_C} = \frac{V_A}{Z_{R_1} + Z_L}$$

Solving for V_A gives us:

$$V_A = (Z_{R_2} + Z_C) \left(Z_{R_1} + Z_L \right) \frac{I_{in} + \frac{V_{in}}{Z_{R_2} + Z_C}}{Z_{R_2} + Z_C + Z_L + Z_{R_1}}$$

We can substitute this value of V_A into our expression for I_1 from Ohm's Law:

$$I_o = \frac{(Z_{R_2} + Z_C)I_{in} + V_{in}}{Z_{R_2} + Z_C + Z_L + Z_{R_1}} = \frac{-j(1-j)+1}{1-j+j+2} = \frac{-j}{3} = \frac{1}{3}e^{-j\pi/2}$$

converting back to the time domain gives us:

$$i_o(t) = \frac{1}{3}\cos(2t - \pi/2) = \frac{1}{3}\sin(2t)$$

Example 5

Let $v(t) = 2 \cos(3t + \pi/6) V$, $i(t) = \cos(3t - \pi/6) A$, $R = 2 \Omega$, C = 1/6 F, and L = 1 H. Determine an equivalent circuit as a voltage source in series with a resistor and an inductor.



Solutions

Since the voltage and current sources are both sinusoidal of the same frequency, we can represent the circuit using phasors and impedances:



To solve for Z_{th} , set voltage sources to 0V (short) and current sources to 0A (open).



$$Z_{th} = Z_R / Z_C + Z_L = Z_L + \frac{Z_R Z_C}{Z_R + Z_C} = 3j + \frac{-4j}{2 - 2j} = 3j - \frac{2j}{1 - j} \left(\frac{1 + j}{1 + j}\right) = 3j - \frac{-2 + 2j}{2}$$
$$= 1 + 2j$$

To solve for V_{th} , leave the output open and solve for V_{th} :



By Ohm's Law:

$$I_R = (V_{th} - V)/Z_R$$

 $I_C = V_{th}/Z_C$
 $I_L = 0$
By KCL:
 $I_R + I_C + I + I_L = 0$

Substituting the expressions for I_R , I_C , and I_L into the KCL equation yields:

$$\frac{V_{th} - V}{Z_R} + \frac{V_{th}}{Z_C} + I = 0$$

Solving for V_{th} gives us:

$$V_{th} = \frac{\frac{V}{Z_R} - I}{\frac{1}{Z_R} + \frac{1}{Z_C}} = \left(\frac{V}{Z_R} - I\right) \left(\frac{Z_C}{Z_R}\right) = \left(\frac{e^{j\pi/6} - e^{-j\pi/6}}{2 - 2j}\right) \frac{-4j}{2 - 2j}$$
$$= \left(\frac{\sqrt{3}}{2} + \frac{j}{2} - \left(\frac{\sqrt{3}}{2} - \frac{j}{2}\right)\right) \left(1 - j\right) = j\left(1 - j\right) = 1 + j = \sqrt{2}e^{j\pi/4}$$

We could have instead solved for the short-circuit current I_{sc} by connecting the terminals (shorting) at the output as follows:



In general, we only need to calculate two out of the three of V_{th} , Z_{th} , and I_{sc} . The third quantity follows from the equation $V_{th} = I_{sc}Z_{th}$. In our case:

$$I_{sc} = \frac{V_{th}}{Z_{th}} = \frac{1+j}{1+2j} = \frac{(1+j)(1-2j)}{5} = \frac{3-j}{5}$$

We know $V_{th} = \sqrt{2} e^{j\pi/4}$ and $Z_{th} = 1 + 2j = R_{th} + j\omega L_{th}$, so converting to the time domain we have

$$R_{th} = \operatorname{Re} \{Z_{th}\} = 1 \Omega$$
$$L_{th} = \frac{\operatorname{Im} \{Z_{th}\}}{\omega} = 2/3 H$$
$$v_{th}(t) = \sqrt{2} \cos(3t + \pi/4)$$

Alternatively, we could a series of use source transformations and series and parallel combinations to solve for Z_{th} and V_{th} ,



$$Z_{th} = Z_L + Z_R / / Z_C = \dots = 1 + 2j$$

 $V_{th} = (V/Z_R - I) (Z_R / / Z_C) = \dots = \sqrt{2}e^{j\pi/4}$