UC San Diego J. Connelly

ECE 45 Discussion 10 Notes

Convolution and Fourier Transform

We have the following relationship between convolution and multiplication in the time and frequency domains:

$$f(t) * g(t) \longleftrightarrow F(\omega)\, G(\omega)$$
$$f(t)\, g(t) \longleftrightarrow \frac{1}{2\pi} F(\omega) * G(\omega)$$

So *convolution* in the **time** domain corresponds to *multiplication* in the **frequency** domain

and *multiplication* in the **time** domain corresponds to *convolution* in the **frequency** domain.

Where convolution is defined by

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy = \int_{-\infty}^{\infty} g(y)\, f(x-y)\, dy$$

Parseval's Theorem for Fourier Transforms

Similar to Parseval's Theorem for the Fourier series, Parseval's Theorem for the Fourier transform tells us that energy is conserved whether we look at it in the time or frequency domain. We may find that the frequency integral is easier to evaluate in some cases.

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega$$

Example 1

Let $y(t) = 2x(t)\cos(10t)$, *where* $\mathcal{F}(x(t)) = \begin{cases} 1 & 1 \leq |\omega| \leq 2 \\ 0 & \textit{else} \end{cases}$

(a) *Determine* $\int_{-\infty}^{\infty} |y(t)|^2\, dt$.

(b) *Is* $y(t)$ *bandlimited?*

Solutions

(a) By the time multiplication property of the Fourier transform

$$Y(\omega) = \frac{1}{2\pi}\, 2\, \mathcal{F}(x(t)) * \mathcal{F}(\cos(10t))$$

Since $\mathcal{F}(1) = 2\pi\delta(\omega)$, by the frequency-shifting property of the Fourier transform, we have

$$\mathcal{F}(\cos(10t)) = \mathcal{F}\left(\frac{e^{j10t} + e^{-j10t}}{2}\right) = \frac{1}{2}\mathcal{F}(e^{j10t}) + \frac{1}{2}\mathcal{F}(e^{-j10t}) = \pi\,\delta(\omega - 10) + \pi\,\delta(\omega + 10).$$

Please report any typos/errors to j2connelly@ucsd.edu

Thus we have

$$Y(\omega) = X(\omega) * (\delta(\omega - 10) + \delta(\omega + 10)) = X(\omega - 10) + X(\omega + 10)$$

$$= \begin{cases} 1 & \substack{8\leq\omega\leq 9 \text{ or} \\ 11\leq\omega\leq 12} \\ 0 & \text{else} \end{cases} + \begin{cases} 1 & \substack{-9\leq\omega\leq -8 \text{ or} \\ -12\leq\omega\leq -11} \\ 0 & \text{else} \end{cases} = \begin{cases} 1 & \substack{8\leq|\omega|\leq 9 \text{ or} \\ 11\leq|\omega|\leq 12} \\ 0 & \text{else} \end{cases}$$

We can use Parseval's Theorem to calculate the energy in the frequency domain

$$\int_{-\infty}^{\infty} |y(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(\omega)|^2 \, d\omega = \frac{1}{2\pi} \left(\int_{-12}^{-11} 1 \, d\omega + \int_{-9}^{-8} 1 \, d\omega + \int_{8}^{9} 1 \, d\omega + \int_{11}^{12} 1 \, d\omega \right) = \frac{2}{\pi}$$

(b) The maximum frequencies present in $y(t)$ are $\omega = \pm 12$, so $y(t)$ is bandlimited.

Example 2

For an LTI system with impulse response $h(t)$, the output is

$$y(t) = \begin{cases} 1 & 3 \leq t \leq 5 \\ 0 & \textit{else} \end{cases}$$

when $f(t)$ is the input. Determine $h(t)$ when

(a) $f(t) = \text{rect}\left(\frac{t}{2} - 1\right)$.

(b) $f(t) = 2u(t)$.

Solutions

(a) Recall for any function $x(t)$ and any real number t_0, $x(t) * \delta(t - t_0) = x(t - t_0)$.

Note that $f(t) = \text{rect}\left(\frac{t-2}{2}\right) = \begin{cases} 1 & 1 \leq t \leq 3 \\ 0 & \text{else} \end{cases}$

$$y(t) = f(t - 2) = f(t) * \delta(t - 2)$$
$$\therefore h(t) = \delta(t - 2)$$

(b)

$$y(t) = 1/2\,[f(t-3) - f(t-5)] = 1/2[\delta(t-3) - \delta(t-5)] * f(t)$$
$$\therefore h(t) = 1/2[\delta(t-3) - \delta(t-5)]$$

Example 3

Evaluate $y(t) = f(t) * g(t)$ *where*

$$f(t) = e^{-a(t-1)}\, u(t-1) \;\text{ and }\; g(t) = t\, u(t) + 1$$

Solutions

Let $h(t) = f(t+1) = e^{-at}\, u(t)$ and let $x(t) = h(t) * g(t)$. By the time-shifting property of convolution:

$$y(t) = f(t) * g(t) = h(t-1) * g(t) = x(t-1)$$

By the distributive property of convolution:

$$x(t) = h(t) * (t\, u(t)) + h(t) * 1$$

$$h(t) * 1 = \int_{-\infty}^{\infty} h(\tau)\, 1\, d\tau = \int_{-\infty}^{\infty} e^{-a\tau}\, u(\tau)\, d\tau = \int_{0}^{\infty} e^{-a\tau} d\tau = \left.\frac{e^{-a\tau}}{-a}\right|_{\tau=0}^{\infty} = \frac{1}{a}$$

$$h(t) * (t\, u(t)) = (t\, u(t)) * h(t) = \int_{-\infty}^{\infty} \tau\, u(\tau)\, h(t-\tau)\, d\tau = \int_{0}^{\infty} \tau\, u(t-\tau)\, e^{-a(t-\tau)}\, d\tau$$

Since τ ranges from 0 to ∞, if $t < 0$, then $u(t-\tau)$ must be 0.
If $t \geq 0$, then $u(t-\tau)$ is 0 if $\tau > t$, so we have

$$= \begin{cases} 0 & \text{for} \quad t < 0 \\ e^{-at} \displaystyle\int_{0}^{t} \tau\, e^{a\tau} d\tau & \text{for} \quad t \geq 0 \end{cases}$$

Using integration by parts, we have:

$$\int_{0}^{t} \tau\, e^{a\tau} d\tau = \left.\frac{\tau\, e^{a\tau}}{a}\right|_{\tau=0}^{t} - \left.\frac{e^{a\tau}}{a^2}\right|_{\tau=0}^{t} = \frac{t\, e^{at}}{a} + \frac{1}{a^2} - \frac{e^{at}}{a^2}$$

So

$$x(t) = \frac{1}{a} + \begin{cases} 0 & \text{for} \quad t < 0 \\ \left(a\, t + e^{-at} - 1\right)/a^2 & \text{for} \quad t \geq 0 \end{cases}$$

Which gives us

$$\begin{aligned} y(t) = x(t-1) &= \begin{cases} 1/a & \text{for} \quad t < 1 \\ 1/a + \big(a\,(t-1) + e^{-a(t-1)} - 1\big)/a^2 & \text{for} \quad t \geq 1 \end{cases} \\ &= \begin{cases} 1/a & \text{for} \quad t < 1 \\ \big(a\, t + e^{-a(t-1)} - 1\big)/a^2 & \text{for} \quad t \geq 1 \end{cases} \end{aligned}$$

Example 4

Let $x(t) = \sin(3t)\, e^{-t}\, u(t)$ *be the input to an LTI system given by the differential equation*

$$\frac{d^2}{dt^2}\, x(t) - \frac{d}{dt}\, y(t) + 10\, x(t) = y(t) - 2\, \frac{d}{dt}\, x(t)$$

Find the output $y(t)$.

Solutions

Taking the Fourier transform of both sides of the differential equation gives us

$$\left((j\omega)^2 + 2j\omega + 10\right)\, X(\omega) = (j\omega + 1)\, Y(\omega)$$

$$\therefore H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{(j\omega)^2 + 2j\omega + 10}{1 + j\omega}$$

The Fourier transform of $x(t)$ is given in the table of FT pairs from the text, but let's do the set up

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} \sin(3t)\, e^{-t}\, e^{-j\omega t}\, dt = \int_{0}^{\infty} \left(\frac{e^{3jt} - e^{-3jt}}{2j}\right)\, e^{-t}\, e^{-j\omega t}\, dt$$

$$\frac{1}{2j} \int_{0}^{\infty} e^{-t(1+j(\omega-3))} + e^{-t(1+j(\omega+3))}\, dt = \ldots = \frac{3}{(1+j\omega)^2 + 9} = \frac{3}{10 + 2j\omega + (j\omega)^2}$$

$$Y(\omega) = X(\omega)\, H(\omega) = \frac{3}{1 + j\omega} \;\longrightarrow\; y(t) = 3\, e^{-t}\, u(t).$$