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ECE 45 Discussion 11 Notes

Impulse Trains

An *impulse train with period* T is the function

$$p(t) = \sum_{n = -\infty}^{\infty} \delta(t - nT)$$

i.e. p(t) are impulses that occur every T seconds. An impulse train is periodic with period T, so we can represent it as its Fourier series

$$p(t) = \sum_{n = -\infty}^{\infty} P_n e^{j\frac{2\pi}{T}nt} \quad \text{where} \quad P_n = \frac{1}{T} \int_T p(t) e^{-j\frac{2\pi}{T}nt} dt$$

In the period [-T/2, T/2], the function $p(t) = \delta(t)$, so

$$P_n = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) \, e^{-j\frac{2\pi}{T}nt} \, dt = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) \, dt = \frac{1}{T}$$

where the second equality follows from the fact $f(t) \delta(t - t_0) = f(t_0) \delta(t - t_0)$ for any function f(t). Hence the Fourier series of an impulse train is

$$p(t) = \frac{1}{T} \sum_{n = -\infty}^{\infty} e^{j\frac{2\pi}{T}nt}$$

which implies the Fourier transform of p(t) can be written as

$$P(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega - n\frac{2\pi}{T}\right)$$

since $\mathcal{F}(e^{j\omega_0 t}) = 2\pi\delta(\omega - \omega_0)$.

Thus the Fourier transform of an impulse train is another impulse train! In time, the impulses are spaced apart by T, whereas in frequency, the impulse are spaced apart by $2\pi/T$.

Sampling a Signal

When we sample a signal, we "pick out" values of the function at certain points. Generally, we do this by taking the value of a function every T_s seconds. i.e. if f(t) is a function, the values of $\dots, f(-T_s), f(0), f(T_s), \dots$ are the sampled values of f(t) with a sampling period of T_s (or equivalently a sampling frequency $\omega_s = 2\pi/T_s$). What is the "best" way to attempt to reconstruct f(t) from its samples?

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Reconstructing a Sampled Signal

It turns out we can *perfectly* reconstruct certain types of signals using some clever math. Suppose we sampled a signal f(t) with period T_s . Let

$$y(t) = \sum_{n=-\infty}^{\infty} f(nT_s) \operatorname{sinc} (nt\pi/T_s)$$

we will show that, in certain cases, the function y(t) is exactly equal to f(t), i.e. this "sinc interpolation" perfectly reconstructs the signal. To see this requires several intermediate steps.

Multiplying by an Impulse Train

Consider the function

$$s(t) = f(t)p(t) = f(t) \sum_{n = -\infty}^{\infty} \delta(t - nT_s) = \sum_{n = -\infty}^{\infty} f(t) \,\delta(t - nT_s) = \sum_{n = -\infty}^{\infty} f(nT_s) \,\delta(t - nT_s)$$

which we call the *sampled signal*. s(t) "picks out" the values of f(t) precisely at the points ..., $-T_s, 0, T_s, ...$ Since s(t) is the product of f(t) and p(t) in time, in frequency, we have

$$S(\omega) = \frac{1}{2\pi}F(\omega) * P(\omega) = \frac{1}{T}F(\omega) * \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} F(\omega - n\omega_s)$$

i.e. $S(\omega)$ is a sum of shifted copies of the Fourier transform of the original signal.

Nyquist Rate

There are two situations that can occur with $S(\omega)$

- 1) the shifted copies of $F(\omega)$ do not interfere/overlap with one another.
- 2) the shifted copies of $F(\omega)$ do intefere/overlap with one another.

When the first case occurs, we can perfectly reconstruct f(t) from s(t) by sending s(t) through a lowpass filter. However, in the second case, perfect reconstruction is not possible, since *aliasing* occurs.

Suppose $F(\omega)$ is *band-limited* by ω_m , i.e. $F(\omega) = 0$ for all $|\omega| < \omega_m$. Then the shifted copies of $F(\omega)$ do not overlap with one another as long as $\omega_m < \omega_s - \omega_m$ or equivalently $\omega_s > 2\omega_m$, i.e. we sample faster the Nyquist rate of f(t).



When $\omega_m > \omega_s/2$, the shifted copies of $F(\omega)$ overlap with one another, which causes aliasing.

Perfect Reconstruction

Now suppose we sample a signal f(t) at a sampling rate $\omega_s > 2\omega_m$, let s(t) be the function from the previous part, and let $H(\omega)$ be the following low-pass filter

$$H(\omega) = \begin{cases} T_s & \text{ if } |\omega| < \omega_s/2 \\ 0 & \text{ otherwise.} \end{cases}$$

Since $\omega_m < \omega_s/2$, the only value of n for which

$$H(\omega) F(\omega - n\omega_s) \neq 0$$

is when n = 0 and so

$$H(\omega) S(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} H(\omega) F(\omega - n\omega_s) = F(\omega).$$

Thus when s(t) is the input to a low-pass filter $H(\omega)$, the output is precisely f(t). Equivalently, in the time domain, we have

$$h(t) = \mathcal{F}^{-1}(H(\omega)) = \operatorname{sinc}(\pi t/T_s)$$

which implies

$$f(t) = h(t) * s(t) = \sum_{n = -\infty}^{\infty} f(nT_s) \operatorname{sinc} (nt\pi/T_s)$$

as claimed.

This process is often modeled graphically by

$$f(t) \longrightarrow \underbrace{s(t)}_{p(t)} F(t) \longrightarrow f(t)$$

Note: this process only perfectly reconstructs f(t) when f(t) is *band-limited* and we sample f's Nyquist rate!

Example

Suppose we perform the following process on a signal f(t)



where
$$p(t) = \sum_{n=-\infty}^{\infty} \delta\left(t - n\frac{\pi}{10}\right)$$
 and $H(\omega) = \begin{cases} \pi/10 & \text{if } |\omega| < 10\\ 0 & \text{otherwise.} \end{cases}$

Determine y(t) when

- (a) $f(t) = \cos(4t)$
- (b) $f(t) = \cos(2t) \cos(4t)$
- (c) $f(t) = \operatorname{sinc}(15t)$

Solutions

This is the reconstruction process detailed above with $\omega_s = 20$. By the previous work we did, we know that if the maximum frequency of $F(\omega)$ is less than 10, this process will produce f(t) at the output. However, when the maximum frequency is greater than 10, it will produce some distorted function.

- (a) When $f(t) = \cos(4t)$, we have $F(\omega) = \pi \delta(\omega 4) + \pi \delta(\omega + 4)$, so $\omega_m = 4$ in this case. Thus y(t) = f(t).
- (b) When $f(t) = \cos(2t) \cos(4t)$, we have

$$F(\omega) = \frac{1}{2\pi} \left(\pi \delta(\omega - 2) + \pi \delta(\omega + 2) \right) * \left(\pi \delta(\omega - 4) + \pi \delta(\omega + 4) \right) = \frac{\pi}{2} \left(\delta(\omega - 6) + \delta(\omega - 2) + \delta(\omega + 2) + \delta(\omega + 2) \right) + \delta(\omega - 2) + \delta(\omega$$

so $\omega_m = 6$ in this case. Thus y(t) = f(t).

Alternatively, $f(t) = \frac{\cos(2t) + \cos(6t)}{2}$ which has maximum frequency equal to 6.

(c) When $f(t) = \operatorname{sinc}(() 15t)$, we have

$$F(\omega) = \begin{cases} \pi/15 & \text{if } |\omega| < 15\\ 0 & \text{otherwise} \end{cases}$$

which implies the shifted copies of $F(\omega)$ in

$$S(\omega) = \frac{10}{\pi} \sum_{n = -\infty}^{\infty} F(\omega - 20n)$$

will interfere with each other. In particular



So when s(t) is the input to the LPF, we have

$$Y(\omega) = S(\omega) H(\omega) = \begin{cases} \pi/15 & \text{if } |\omega| < 5\\ 2\pi/15 & \text{if } 5 < |\omega| < 10\\ 0 & \text{otherwise} \end{cases}$$

One way to take the inverse Fourier transform of this function is to note

$$Y(\omega) = \begin{cases} 2\pi/15 & \text{if } |\omega| < 10\\ 0 & \text{otherwise} \end{cases} - \begin{cases} \pi/15 & \text{if } |\omega| < 5\\ 0 & \text{otherwise} \end{cases}$$

which implies

$$y(t) = \frac{4}{3}\operatorname{sinc}(10t) - 3\operatorname{sinc}(5t) \neq f(t)$$