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ECE 45 Discussion 2 Notes

Frequency Response

The inputs and outputs of RLC circuits are generally either voltages or currents. The output of the circuit depends on the frequency of the input.

- An input/output RLC circuit is defined by its *frequency response* (also called the transfer function): $H(\omega) = \frac{Out(\omega)}{In(\omega)}$, where $Out(\omega)$ and $In(\omega)$ are the phasor transforms of $out(t)$ and $in(t)$, for an arbitrary frequency ω. The inputs and outputs of RLC circuits are generally either voltages or currents.

- In general, $out(t) \neq in(t)\, H(\omega)$ (a common mistake in this course).

- $H(\omega)$ is a complex number, which is a function of ω, since input/output relationships change with different frequencies.

$$|H(\omega)| = \sqrt{\mathsf{Re}\left\{H(\omega)\right\}^2 + \mathsf{Im}\{H(\omega)\}^2} \text{ and } \angle H(\omega) = \tan^{-1}\left(\frac{\mathsf{Im}\{H(\omega)\}}{\mathsf{Re}\left\{H(\omega)\right\}}\right).$$

- Suppose $x(t)$ is sinusoidal, $x(t) = A\,\cos(\omega_0 t + \phi)$, so its phasor representation is $Ae^{j\phi}$. If $x(t)$ is the input to an RLC circuit with frequency response $H(\omega)$, then the output phasor is $Y = A\,|H(\omega_0)|\, e^{j(\phi + \angle H(\omega_0))}$, so

$$y(t) = A\,|H(\omega_0)|\,\cos(\omega_0\, t + \phi + \angle H(\omega_0))$$

  The frequency response describes how the system *responds* to a sinusoid of a particular *frequency*.

- When $x(t) = \sum A_i\,\cos(\omega_i + \phi_i)$, by super position, the output is

$$y(t) = \sum A_i\,|H(\omega_i)|\,\cos(\omega_i + \phi_i + \angle H(\omega_i))$$

- This allows us to analyze RLC circuits whose inputs are voltages and currents consisting of multiple frequencies.

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Example 1

Find the frequency $H(\omega)$ of an RLC circuit, with input $x(t)$ and output $y(t)$, given by

$$\frac{d^2y(t)}{dt^2} + x(t) = \frac{dx(t)}{dt} + y(t).$$

Determine the output when the input $x(t) = \cos(2t - \pi/4)$.

Solutions

Assume $x(t)$ is sinusoidal, then we can represent the differential equation using phasors:

$$(j\omega)^2 Y + X = j\omega X + Y$$

and so

$$H(\omega) = \frac{Y}{X} = \frac{j\omega - 1}{(j\omega)^2 - 1} = \frac{1}{j\omega + 1}.$$

When $x(t) = \cos(2t)$, the output is given by

$$y(t) = |H(2)| \cos(2t - \pi/4 + \angle H(2))$$

where $|H(2)| = \dfrac{1}{\sqrt{5}}$ and $\angle H(2) = \tan^{-1}(2)$.

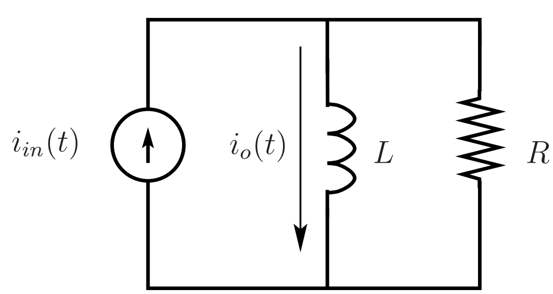
Example 2

(a) *Find the frequency response, $H(\omega)$, of the circuit below with input $i_{in}(t)$ and output $i_o(t)$.*

(b) *Find the value ω_0 for which $|H(\omega)|$ is maximized.*

(c) *Find the value $\omega_1 > 0$ such that $|H(\omega_1)| = |H(\omega_0)|/\sqrt{2}$*

(d) *With the value of R fixed, what value should L take so that we have $|H(10^4)| = |H(\omega_0)|/\sqrt{2}$*



$$R = 100\,k\Omega,\ \ L = 1\,mH$$

Solutions

(a) We can use a current divider to find the current through the inductor:

$$I_o = I_i \frac{\frac{1}{Z_L}}{\frac{1}{Z_R} + \frac{1}{Z_L}} = \frac{1}{\frac{Z_L}{Z_R} + 1} \quad \text{and so} \quad H(\omega) = \frac{I_o}{I_i} = \frac{1}{1 + j\omega(L/R)}$$

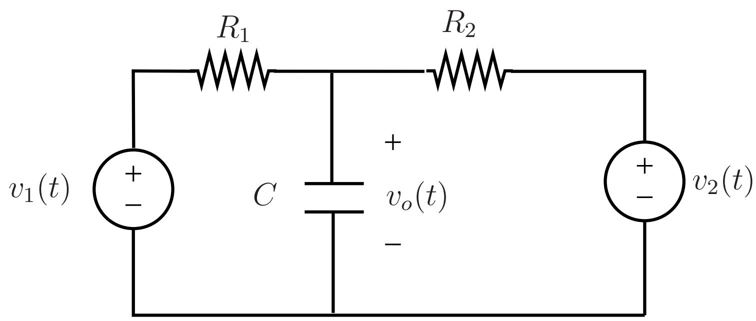
(b) To maximize $|H(\omega)|$ with respect to ω, note that

$$|H(\omega)| = \frac{1}{\sqrt{1+(\omega L/R)^2}} \quad \rightarrow \quad |H|_{max} = |H(0)| = 1$$

(c) $|H(\omega_1)| = \dfrac{1}{\sqrt{1+(\omega_1 L/R)^2}} = \frac{1}{\sqrt{2}} \longrightarrow 1+(\omega_1 L/R)^2 = 2 \longrightarrow \omega_1 = R/L = 10^5/10^{-3} = 10^8$

1. $10^4 = \omega_1 = R/L = 10^5/L \rightarrow L = 10H$

Example 3



$R_1 = 2\Omega$, $R_2 = 1\Omega$, and $C = 1/2F$.

(a) *When $v_2(t) = 0$ and $v_1(t)$ is the input to the circuit, find the frequency response $H_1(\omega) = V_o/V_1$.*

(b) *When $v_1(t) = 0$ and $v_2(t)$ is the input to the circuit, find the frequency response $H_2(\omega) = V_o/V_2$.*

(c) *Find the output $v_o(t)$ when $v_1(t) = 2\cos(3t + \pi/3)$ and $v_2(t) = 3\sin(2t)$*

Solutions

(a) When $v_2(t) = 0$, by using a voltage divider on the phasor-transformed circuit, we have

$$V_o = V_1 \frac{(R_2 // \frac{1}{j\omega C})}{R_1 + (R_2 // \frac{1}{j\omega C})} = V_1 \frac{R_2}{R_1 + R_2 + j\omega C R_1 R_2}$$

and so

$$H_1(\omega) = \frac{V_o}{V_1} = \frac{1}{3 + j\omega} = \frac{1}{\sqrt{9+\omega^2}} e^{-j\tan^{-1}(\omega/3)}$$

(b) Similarly, when $v_1(t) = 0$, by using a voltage divider on the phasor-transformed circuit, we have

$$V_o = V_2 \frac{(R_1 // \frac{1}{j\omega C})}{R_2 + (R_1 // \frac{1}{j\omega C})} = V_2 \frac{R_1}{R_1 + R_2 + j\omega C R_1 R_2}$$

and so

$$H_2(\omega) = \frac{V_o}{V_2} = \frac{2}{3 + j\omega} = \frac{2}{\sqrt{9+\omega^2}} e^{-j\tan^{-1}(\omega/3)}$$

(c) We know that when $v_2(t) = 0$ and $v_1(t) = 2\cos(3t + \pi/3)$, the output is

$$v_{o,1}(t) = 2|H_1(3)|\cos(3t + \pi/3 + \angle H_1(3))$$

and when $v_1(t) = 0$ and $v_2(t) = 3\sin(2t)$, the output is

$$v_{o,2}(t) = 3|H_2(2)|\sin(2t + \angle H_2(2)).$$

So by super-position, the output is

$$v_o(t) = v_{o,1}(t) + v_{o,2}(t) = \frac{2}{\sqrt{18}}\cos(3t + \pi/3 - \tan^{-1}(1) + \frac{6}{\sqrt{13}}\sin(2t - \tan^{-1}(2/3))$$
$$= \frac{\sqrt{2}}{3}\cos(3t + \pi/12) + \frac{6}{\sqrt{13}}\sin(2t - \tan^{-1}(2/3))$$

Example 4

Suppose the output of an RLC circuit is

$$\frac{1}{\sqrt{16+\omega^2}}\cos(\omega t - \tan^{-1}(\omega/4))$$

when the input is $\cos(\omega t)$*, where* $\omega \geq 0$*.*

(a) *What is the output when the input is* $A\cos(\omega t + \theta)$ *for some real numbers* A, θ*?*

(b) *What is the output when the input is* $2 + \sin(4t)$*?*

Solutions

(a) When the input to an RLC circuit is $\cos(\omega t)$, the output is

$$|H(\omega)|\cos(\omega t + \angle H(\omega)).$$

Hence

$$H(\omega) = \frac{e^{-j\tan^{-1}(\omega/4)}}{\sqrt{16+\omega^2}}$$

and the output when $A\cos(\omega t + \theta)$ is

$$\frac{A}{\sqrt{16+\omega^2}}\cos(\omega t + \theta - \tan^{-1}(\omega/4)).$$

This is a useful trick to use in general. That is, we can find the output when $\cos(\omega t)$ is the input, then apply any scaling and shifting after the fact. This can simplify some of the phasor analysis of circuits. This utilizes the *linearity* and *time invariance* of an RLC circuit.

(b) Let $x_1(t) = 2$ and $x_2(t) = \sin(4t)$, and suppose $y_1(t)$ and $y_2(t)$ are the outputs when $x_1(t)$ and $x_2(t)$, respectively, are the inputs. Then by super position, when $x_1(t) + x_2(t)$ is the input, $y_1(t) + y_2(t)$ is the output.

$x_1(t)$ is the case in (a), where $A = 2$, $\omega = 0$, and $\theta = 0$, so $y_1(t) = \frac{2}{4}$.

$x_2(t)$ is the case in (b), where $A = 1$, $\omega = 4$, and $\theta = -\pi/2$, so

$$y_2(t) = \frac{1}{\sqrt{32}} \cos(4t - \pi/2 - \tan^{-1}(1)).$$

Thus the desired output is

$$\frac{1}{2} + \frac{1}{4\sqrt{2}} \sin(4t - \pi/4)$$

Example 5

For an RLC circuit with frequency response

$$H(\omega) = \begin{cases} 1 - 2j\omega & |\omega| \leq 1/2 \\ 0 & \textit{otherwise} \end{cases}$$

find the output $y(t)$ when the input is

$$x(t) = \sum_{k=0}^{\infty} \frac{1}{1+k} \cos(kt/3).$$

Solutions

For $k = 0, 1, 2, \ldots$, let $x_k(t) = \frac{1}{1+k} \cos(kt/3)$, and let $y_k(t)$ be the output of the system when $x_k(t)$ is the input. Then, since $H(\omega)$ is the frequency response of an RLC circuit,

$$y_k(t) = |H(k/3)| \frac{1}{1+k} \cos(kt/3 + \angle H(k/3))$$

and we have $|H(k/3)| = 0$ for $k \geq 2$, and $H(0) = 1$, and $H(1/3) = 1 - 2j/3 = \sqrt{1 + 4/9}\, e^{j \tan^{-1}(-2/3)}$, so

$$y_0(t) = 1 \text{ and } y_1(t) = \frac{\sqrt{7}}{6} \cos(t/3 - \tan^{-1}(2/3))$$

By super-position/linearity, we have

$$y(t) = \sum_{k=0}^{\infty} y_k(t) = 1 + \frac{\sqrt{7}}{6} \cos(t/3 - \tan^{-1}(2/3)).$$