UC San Diego J. Connelly

ECE 45 Discussion 3 Notes

Bode Plots:

Recall the *frequency response* $H(\omega)$ of an LTI system is such that

$$A\cos(\omega_o t + \theta) \longrightarrow H(\omega) \longrightarrow A|H(\omega_o)|\cos(\omega_o t + \angle H(\omega_o) + \theta).$$

$H(\omega)$ represents the change in magnitude and the change in phase when the input to the system is sinusoidal with frequency ω.

Bode plots are a ways of plotting and approximating the *magnitude* $|H(\omega)|$ and *phase* $\angle H(\omega)$ for a very large range of frequencies without losing precision.

We generally plot the magnitude in decibels: $20\,\log_{10}(|H(\omega)|)$ and the phase in radians. In each plot the ω-axis is on a logarithmic scale, i.e. the "ticks" are powers of 10.

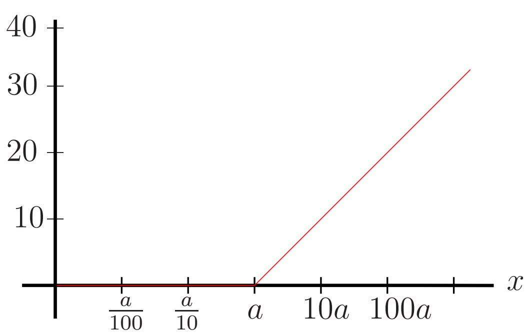
Bode plots make use of two mathematical approximations that are fairly accurate when considering orders of magnitude changes in x for $x, a > 0$:

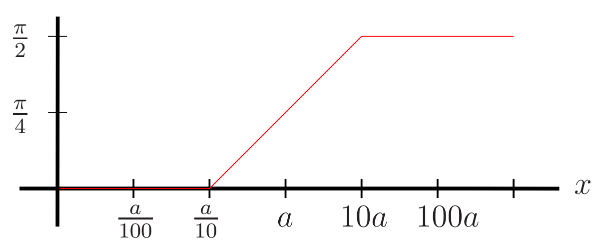
$$\log_{10}\left(\sqrt{1+\left(\frac{x}{a}\right)^2}\right) \approx \begin{cases} \log_{10}(x/a) & x \geq a \\ 0 & x < a \end{cases}$$

$$\tan^{-1}\left(\frac{x}{a}\right) \approx \begin{cases} 0 & x < \frac{a}{10} \\ \pi/4\,\log_{10}\left(\frac{10x}{a}\right) & \frac{a}{10} \leq x < 10a \\ \pi/2 & x \geq 10a \end{cases}$$

Both of these are approximately linear (in certain regions) on a log scale. i.e.:

$$\log_{10}\left(\sqrt{1+\left(\frac{x}{a}\right)^2}\right) \approx \qquad\qquad \tan^{-1}\left(\frac{x}{a}\right) \approx$$





We can use these approximations to plot the magnitude and phase of a transfer function $H(\omega)$.

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Suppose we have a transfer function

$$H(\omega) = \frac{x_0 + x_1(j\omega) + x_2(j\omega)^2 + \cdots + x_m(j\omega)^m}{y_0 + y_1(j\omega) + y_2(j\omega)^2 + \cdots + y_n(j\omega)^n}$$

where $x_0, \ldots, x_m$ and $y_0, \ldots, y_1$ are complex numbers. (Nearly all transfer functions we encounter are of this form). By factoring polynomials, we can write $H(\omega)$ in *standard form*:

$$H(\omega) = C\,\omega^k \frac{\left(1 + \frac{j\omega}{a_1}\right) \cdots \left(1 + \frac{j\omega}{a_r}\right)}{\left(1 + \frac{j\omega}{b_1}\right) \cdots \left(1 + \frac{j\omega}{b_s}\right)}$$

where $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$ are real numbers, C is some constant complex number, and k is some positive or negative integer. Having a transfer function in *standard form* allows us to use our earlier approximations, since we can write each linear term in polar form as

$$1 + \frac{j\omega}{a} = \sqrt{1 + \left(\frac{\omega}{a}\right)^2}\, e^{j\tan^{-1}\left(\frac{\omega}{a}\right)}$$

Approximating the magnitude of a transfer function in standard form:

The magnitude of a product of complex numbers is the product of the magnitudes (i.e. $|XY| = |X||Y|$), and the log of a product is the sum of the logs (i.e. $\log(AB) = \log(A) + \log(B)$). Thus

$$20\,\log_{10}\left(|H(\omega)|\right) = 20\left(\log_{10}(|C|) + k\log_{10}(\omega) + \sum_{n=1}^{r}\log_{10}\left(\sqrt{1 + \left(\frac{\omega}{a_n}\right)^2}\right) - \sum_{n=1}^{s}\log_{10}\left(\sqrt{1 + \left(\frac{\omega}{b_n}\right)^2}\right)\right)$$

$$20\,\log_{10}\left(\sqrt{1 + \left(\tfrac{\omega}{a_n}\right)^2}\right) \approx \begin{cases} 0 & \text{when } \omega < a_n \\ \text{adds 20 dB/dec} & \text{when } \omega \geq a_n \end{cases}$$

$20\,\log_{10}(|C|)$is present at all frequencies.

$20k\,\log_{10}(\omega)$ is zero when $\omega = 1$ and adds $20k$ dB/dec.

Approximating the phase of a transfer function in standard form:

The phase of a product of complex numbers is the sum of the phases (i.e. $\angle XY = \angle X + \angle Y$). Thus

$$\angle H(\omega) = \angle(C) + \angle(\omega^k) + \sum_{n=1}^{r}\tan^{-1}\left(\frac{\omega}{a_n}\right) - \sum_{n=1}^{s}\tan^{-1}\left(\frac{\omega}{b_n}\right)$$

$\angle(C)$ is present at all frequencies, and ω^k is a real number, so $\angle(\omega^k) = 0$.

$$\tan^{-1}\left(\tfrac{\omega}{a_n}\right) \approx \begin{cases} 0 & \text{when} \quad \omega < a_n/10 \\ \text{adds } \pi/4 \text{ rads/dec} & \text{when} \quad a_n/10 \leq \omega < 10a_n \\ \pi/2 & \text{when} \quad \omega \geq 10a_n \end{cases}$$

Approximating Values from the Bode Plot:

We can use the fact Bode plots are a logrithmic linear approximation of the magnitude and phase to find the values of points:

$$\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{\log(x_2) - \log(x_1)} = \frac{y_2 - y_1}{\log(x_2/x_1)}.$$

Example 1:

Draw the Bode plot (amplitude and phase) and find all critical points of the transfer function

$$H(\omega) = \frac{j\omega}{1000 + j\omega}$$

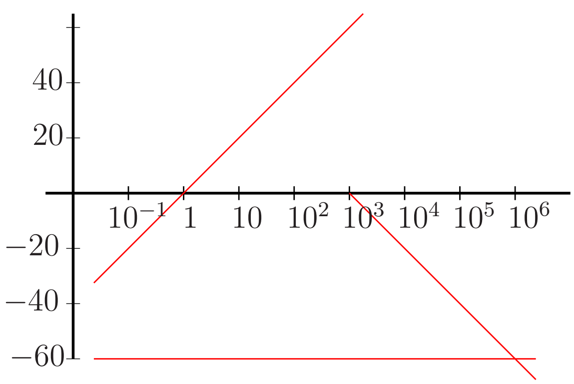
$H(\omega)$ is a simple high pass filter. We first write $H(\omega)$ in standard form:

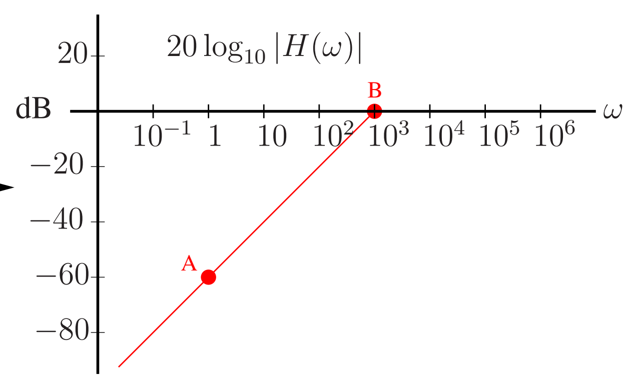
$$H(\omega) = \frac{j\omega}{1000\left(1 + \frac{j\omega}{1000}\right)}$$

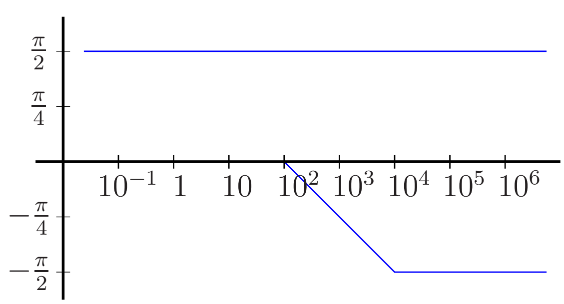
Then

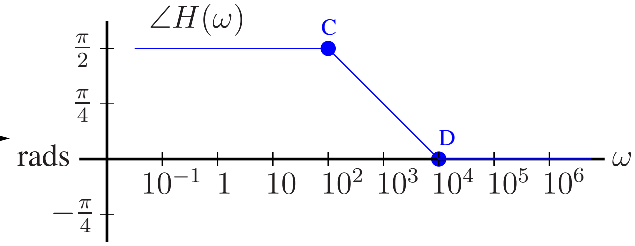
$$20\log_{10}\left(|H(\omega)|\right) = 20\log_{10}(\omega) - 60 - 20\log_{10}\left(\sqrt{1 + \left(\frac{\omega}{1000}\right)^2}\right)$$

$$\angle H(\omega) = \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega}{10^3}\right).$$









To find the height of point A, we note $20\log_{10}(1) = 0$, so -60 is the only "active" term at $\omega = 1$.

To find the height of point B, we use the fact the slope is 20 dB per decade. So the height of B is

$$-60 + 20(\log_{10}(10^3) - \log_{10}(1)) = 0.$$

We also have C $= (100, \pi/2)$ and D $= (10^4, 0)$.

Example 2:

Draw the Bode plot (amplitude and phase) and find all critical points of the transfer function

$$H(\omega) = \frac{500 + 5j\omega}{(1 + j\omega)\left(10 + \frac{j\omega}{100}\right)}$$

We first write $H(\omega)$ in standard form:

$$H(\omega) = \frac{50\left(1 + \frac{j\omega}{100}\right)}{(1 + j\omega)\left(1 + \frac{j\omega}{1000}\right)}$$

Then

$$\begin{aligned} 20\log_{10}(|H(\omega)|) = 20 + 20\log_{10}(5) + 20\log_{10}\left(\sqrt{1 + \left(\frac{\omega}{100}\right)^2}\right) \\ - 20\log_{10}\left(\sqrt{1+\omega^2}\right) - 20\log_{10}\left(\sqrt{1 + \left(\frac{\omega}{1000}\right)^2}\right) \end{aligned}$$

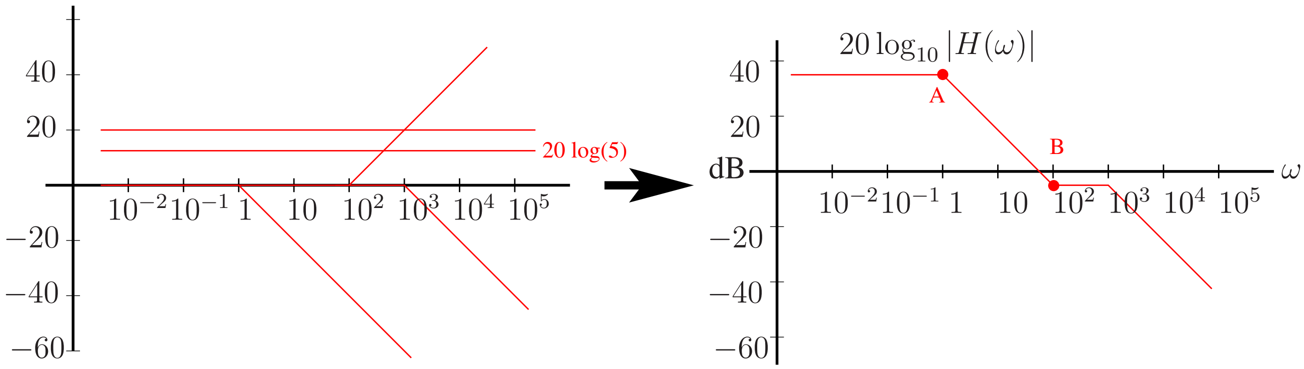
$$\angle H(\omega) = \tan^{-1}\left(\frac{\omega}{100}\right) - \tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{1000}\right).$$

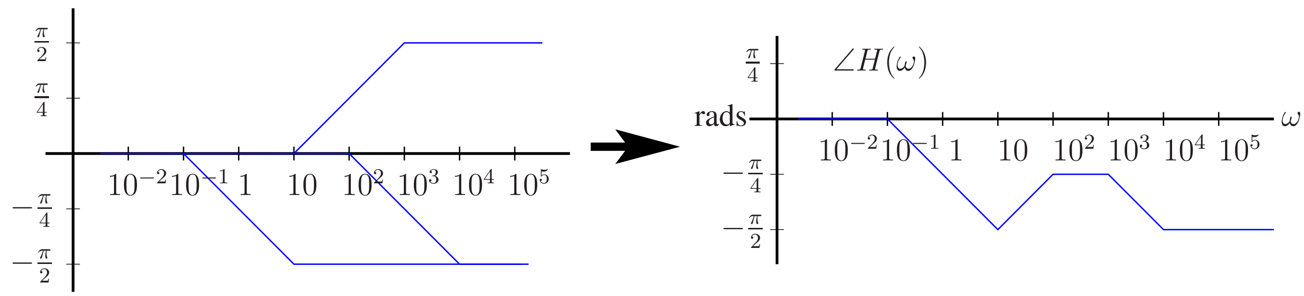
The height at point A is $20(1 + \log_{10}(5))$.

The height at point B is $-20 + 20\log_{10}(5)$, since the slope is -20 dB/decade and the frequency changes by a factor of 100.

$20\log_{10}(|H(\omega_0)|) = 0$ when

$$\begin{aligned} 0 &= 20(1 + \log_{10}(5)) - 20(\log_{10}(\omega_0) - \log_{10}(1)) \\ &\rightarrow\ 1 + \log_{10}(5) = \log_{10}(\omega_0)\ \rightarrow\ \omega_0 = 50 \end{aligned}$$





Example 3:

Draw the Bode plot (amplitude and phase) and find all critical points of the transfer function

$$H(\omega) = \frac{3\,(400 - \omega^2 + 40j\omega)}{2\,(j\omega + 6000)\,(j\omega\,10^{-6} + 1)}$$

We first write $H(\omega)$ in standard form:

$$H(\omega) = \frac{(\frac{j\omega}{20} + 1)^2}{10\left(\frac{j\omega}{6000} + 1\right)\left(\frac{j\omega}{10^6} + 1\right)}$$

Then

$$20\log_{10}(|H(\omega)|) = 40\log_{10}\left(\sqrt{1 + \left(\frac{\omega}{20}\right)^2}\right) - 20 - 20\log_{10}\left(\sqrt{1 + \left(\frac{\omega}{6000}\right)^2}\right) - 20\log_{10}\left(\sqrt{1 + \left(\frac{\omega}{10^6}\right)^2}\right)$$

$$\angle H(\omega) = 2\tan^{-1}\left(\frac{\omega}{20}\right) - \tan^{-1}\left(\frac{\omega}{6000}\right) - \tan^{-1}\left(\frac{\omega}{10^6}\right).$$

The change in $\log(\omega)$ from point A to B is $\log(6000) - \log(20)$, and the slope (dB per decade) in this region is 40, so the height at B is

$$-20 + 40(\log(6000) - \log(20)) = 60 + 40\log(3).$$

The height at C is

$$(60 + 40\log(3)) + 20(\log(10^6) - \log(6000)) = 120 + 20\log(3) - 20\log(2)$$

$20\ \log_{10}(|H(\omega_0)|) = 0$ when

$$0 = -20 + 40(\log(\omega_0) - \log(20)) \longrightarrow \omega_0 = 20\sqrt{10} \approx 63.2$$

We also have

$$D = (2, 0),\ E = (200, \pi),\ F = (600, \pi),\ G = (60000, \pi/2),\ H = (10^5, \pi/2),\ I = (10^7, 0)$$