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ECE 45 Discussion 4 Notes

LTI Systems

A *system* has a time-domain input and a time-domain output.

$$x(t) \longrightarrow \text{[System]} \longrightarrow y(t)$$

A system is *linear* if it satisfies the following:

If

$$x_1(t) \longrightarrow \text{[Linear System]} \longrightarrow y_1(t)$$
$$x_2(t) \longrightarrow \text{[Linear System]} \longrightarrow y_2(t)$$

then for all complex numbers a and b

$$a\,x_1(t) + b\,x_2(t) \longrightarrow \text{[Linear System]} \longrightarrow a\,y_1(t) + b\,y_2(t)$$

i.e. if the input is a linear combination of functions, the the output is the same linear combination of the outputs corresponding to those functions.

A system is *time-invariant* if it satisfies the following:

If

$$x(t) \longrightarrow \text{[Time-Invariant System]} \longrightarrow y(t)$$

then

$$x(t - t_0) \longrightarrow \text{[Time-Invariant System]} \longrightarrow y(t - t_0).$$

i.e. if the input is delayed by some amount, the output is delayed by the same amount.

A system *LTI* if it is both linear and time-invariant. An LTI system is defined by its frequency response $H(\omega)$, i.e. how a system scales the magnitude and time-shifts a sinusoid of frequency ω. In particular, complex exponentials are *eigenfunctions* of LTI systems. That is:

$$e^{j\omega_0 t} \;\rightarrow\; H(\omega) \;\rightarrow\; H(\omega_0)\,e^{j\omega_0 t} = |H(\omega_0)|\,e^{j(\omega_0 t + \angle H(\omega_0))}$$

If we send a sum of scaled exponentials into an LTI system, since the system is linear, we have:

$$\sum_k A_k\,e^{j\omega_k t} \;\rightarrow\; H(\omega) \;\rightarrow\; \sum_k H(\omega_k)\,A_k\,e^{j\omega_k t}$$

Note that in general, $y(t) \neq H(\omega)x(t)$. This is a common mistake. $y(t) = H(\omega_0)x(t)$, when $x(t)$ is a complex exponential, but this is a special case.

If we could represent an arbitrary input as a **sum of complex exponentials**, then the output would be the sum of the outputs of the individual exponentials.

The Fourier Series allows us to do exactly that for *periodic functions*.

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Periodic Functions

A periodic function is one which repeats itself every fixed amount of time. The *period* T is the minimum amount of time in which the function repeats itself. Every periodic function has a *fundamental frequency*, $\omega_0 = 2\pi/T$. Formally, a function is periodic if there exists $\tau > 0$ such that $f(t) = f(t + \tau)$ for all t, and the period T is the minimum such τ.

Fourier Series

In order to represent a periodic function we need to know two things:

1) The fundamental frequency of the function,

2) How that function behaves over one period.

With these two pieces of information, we can decompose the function into a sum of complex exponentials, where the frequency of each exponential is a multiple of the fundamental frequency.

For a periodic function, $f(t)$, its Fourier Series representation is:

$$f(t) = \sum_{n=-\infty}^{\infty} F_n \, e^{jn\omega_0 t}$$

$$\text{where } F_n = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) \, e^{-jn\omega_0 t} \, dt$$

Note: $\omega_0 = 2\pi/T$, and t_0 is ANY time. Usually it is convenient to pick $t_0 = 0$ (depends on function).

It is helpful to think of the integral as "filtering out" any portion of the signal except the contribution of the sinusoidal function at frequency $n\,\omega_0$. F_n is how much the sinusoidal frequency at $\omega_0\,n$ contributes to the signal. Thus summing over all n will yield the signal itself.

Example 1

*We will say a system is **time-scaling invariant** if it satisfies the following: If*

$$x(t) \longrightarrow [\textit{time-scaling invariant system}] \longrightarrow y(t)$$

then for any real number a,

$$x(at) \longrightarrow [\textit{time-scaling invariant system}] \longrightarrow y(at)$$

Are LTI systems necessarily time-scaling invariant?

Solutions

Consider a system where the output is the derivative of the input. We can verify that this is a linear and time-invariant system. Let $x_1(t)$ and $x_2(t)$ be arbitrary functions and, for each $k = 1, 2$, suppose $y_k(t)$ is the output of our system when $x_k(t)$ is the input. Then for any real numbers a, b, c, d, suppose the input to the system is $ax_1(t-b) + cx_2(t-d)$. Then the output of the system is

$$\frac{d}{dt}\left(ax_1(t-b) + cx_2(t-d)\right) = a\frac{d}{dt}x_1(t-b) + c\frac{d}{dt}x_2(t-d) = ay_1(t-b) + cy_2(t-d).$$

Thus this system is both linear and time-invariant. In fact, the frequency response of such a system is $H(\omega) = j\omega$. We will now show this LTI system is **not** time-scaling invariant.

For an arbitrary function $x(t)$, suppose $y(t) = \frac{d}{dt}x(t)$ is the output when $x(t)$ is the input. Now if the input to this system is $x(2t)$, then by the chain rule of derivatives, the output of the system is

$$2\frac{d}{dt}x(2t).$$

However, this is not equal to $y(2t) = \frac{d}{dt}x(2t)$, so we have demonstrated an LTI system which is **not** time-scaling invariant.

Alternatively, consider an LTI system (an ideal LPF) with frequency response

$$H(\omega) = \begin{cases} 1 & |\omega| < 1 \\ 0 & \text{else} \end{cases}$$

Then $\cos(t/2)$ is the output when $\cos(t/2)$ is input, and 0 is the output when $\cos(t)$ is the input. Clearly $0 \neq \cos(t)$, so this system is also not time-scaling invariant.

Example 2

Are the following systems linear? Are they time invariant?

(a) $x(t) \longrightarrow$ [System (a)] $\longrightarrow 5x(t-10)$

(b) $x(t) \longrightarrow$ [System (b)] $\longrightarrow (x(t)+t)^2$

(c) $x(t) \longrightarrow$ [System (c)] $\longrightarrow x(t)+1$

(d) $x(t) \longrightarrow$ [System (d)] $\longrightarrow \cos(x(t))$

(e) $x(t) \longrightarrow$ [System (e)] $\longrightarrow \int_{-\infty}^{t} x(\tau)\, d\tau$

i.e. the term on the right is the output when the input is $x(t)$.

Solutions

(a) For any functions $x_1(t), x_2(t)$ and real numbers a, b, t_1, t_2, we have

$$ax_1(t-t_1) + bx_2(t-t_2) \longrightarrow \text{[System (a)]} \longrightarrow a5x_1(t-t_1-10) + b5x_2(t-t_2-10)$$

Thus the system is both linear and time invariant.

(b) For any function $x(t)$ and any real number a, we have

$$ax(t) \longrightarrow \text{[System (b)]} \longrightarrow (ax(t)+t)^2 \neq a\,(x(t)+t)^2$$

so the system is not linear.

For any real number t_0, we have

$$x(t-t_0) \longrightarrow \text{[System (b)]} \longrightarrow (x(t-t_0)+t)^2 \neq (x(t-t_0)+t-t_0)^2$$

so the system is not time invariant.

(c) For any function $x(t)$, we have

$$x(t) - x(t) = 0 \longrightarrow \text{[System (c)]} \longrightarrow 1 \neq 0$$

so the system is not linear.

For any real number t_0, we have

$$x(t-t_0) \longrightarrow \text{[System (c)]} \longrightarrow x(t-t_0)+1$$

so the system is time invariant.

(d) For any functions $x(t)$, we have

$$x(t) - x(t) = 0 \longrightarrow \text{[System (d)]} \longrightarrow \cos(0) = 1 \neq 0$$

so the system is not linear.

For any real number t_0, we have

$$x(t-t_0) \longrightarrow \text{[System (d)]} \longrightarrow \cos(x(t-t_0))$$

so the system is time invariant.

(e) Note that for any function $x(t)$ and any real number c, by letting $z = \tau - c$, we have

$$\int_{-\infty}^{t} x(\tau - c)\, d\tau = \int_{-\infty}^{t-c} x(z)\, dz$$

For any functions $x_1(t), x_2(t)$ and real numbers a, b, t_1, t_2, we have

$$ax_1(t - t_1) + bx_2(t - t_2) \longrightarrow [\text{ System (e) }] \longrightarrow a\int_{-\infty}^{t} x_1(\tau - t_1)\, d\tau + b\int_{-\infty}^{t} x_2(\tau - t_1)\, d\tau$$
$$= a\int_{-\infty}^{t-t_1} x_1(z)\, dz + b\int_{-\infty}^{t-t_2} x_2(z)\, dz.$$

Thus the system is both linear and time invariant.

Example 3

Find the fundamental frequency ω_0 and the period T of the following functions:

(a) $f_1(t) = \sin(2t) + 2\cos(3t + \pi/4) - \cos(t/2)$

(b) $f_2(t) = \sum_{n=-\infty}^{\infty} x(t - 3\,n)$ where $x(t) = \begin{cases} 0 & t < 0 \text{ or } t > 3 \\ t & 0 < t < 1 \\ 1 & 1 < t < 3 \end{cases}$

Solutions

(a) We need to find a minimal time interval in which each term in $f_a(t)$ starts/ends a cycle.
The period of $\sin(2t)$ is π. So it starts/ends a cycle at $0, \pi, 2\pi, 3\pi, 4\pi, \ldots$
The period of $2\cos(3t + \pi/4)$ is $2\pi/3$. So it starts/ends a cycle at $0, 2\pi/3, 4\pi/3, 2\pi, 8\pi/3, \ldots$
The period of $\cos(t/2)$ is 4π. So it starts/ends a cycle at $0, 4\pi, 8\pi, \ldots$
Each term starts a cycle at $t = 0$ and ends a cycle at $t = 4\pi$.
Thus $f_1(t)$ is periodic with period $T = 4\pi$, so $\omega_0 = 1/2$.
To verify this is correct, note that

$$f_1(t - 4\pi) = \sin(2t - 8\pi) + 2\cos(3t + \pi/4 - 12\pi) - \cos(t/2 - 2\pi)$$
$$= \sin(2t) + 2\cos(3t + \pi/4) - \cos(t/2) = f_1(t).$$

(b) $f_2(t)$ is a way of writing a periodic function with period $T = 3$, so $\omega_0 = 2\pi/3$.
To verify this is correct, note that

$$f_2(t - 3) = \sum_{n=-\infty}^{\infty} x(t - 3n - 3) = \sum_{n=-\infty}^{\infty} x(t - 3(n + 1)) = \sum_{k=-\infty}^{\infty} x(t - 3k) = f_2(t).$$

Example 4

Find the Fourier series components F_n of $f(t) = \sin^4(t)$.

Solutions

We could use the standard method of integrating $f(t)e^{-j\omega_0 nt}$ in a period to find F_n; however, using Euler's formula, we have

$$\begin{aligned}\sin^4(t) &= \left(\frac{e^{jt} - e^{-jt}}{2j}\right)^4 = \frac{1}{16}\left(\left(e^{jt} - e^{-jt}\right)^2\right)^2 \\ &= \frac{1}{16}\left(e^{2jt} + e^{-2jt} - 2\right)^2 \\ &= \frac{1}{16}\left(e^{4jt} + e^{-4jt} - 4e^{2jt} - 4e^{-2jt} + 6\right).\end{aligned}$$

Thus $\omega_0 = 2$ and

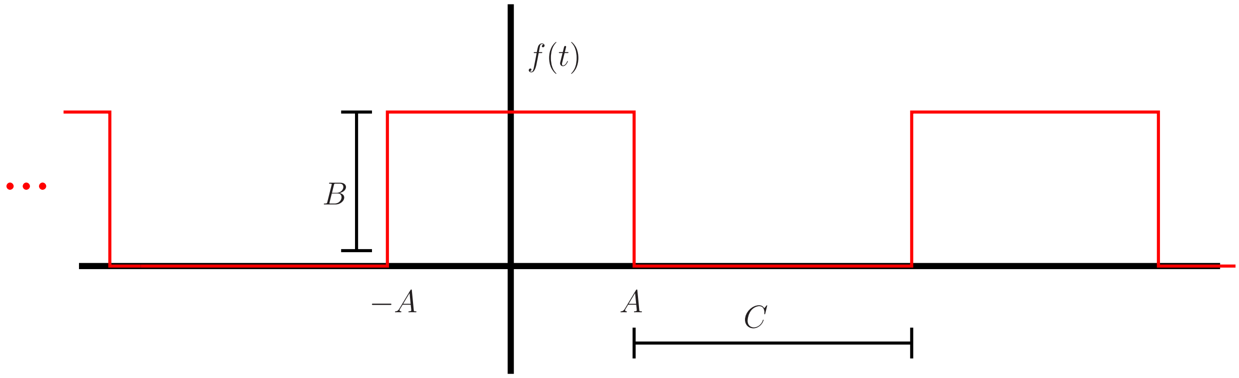
$$F_n = \begin{cases} 1/16 & n = \pm 2 \\ -1/4 & n = \pm 1 \\ 3/8 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

and in fact

$$f(t) = \frac{1}{8}\left(\cos(4t) - 4\cos(2t) + 3\right)$$

Example 5

Find the Fourier Series components F_n of the periodic function $f(t)$, where $A, B, C > 0$.



Solutions

In order to represent $f(t)$ as a Fourier series, we need its period and a mathematical expression for its behavior in a period.

$$T = 2A + C \rightarrow \omega_0 = \frac{2\pi}{2A + C}$$

since $f(t + (2A + C)) = f(t)$ for all t

Over a period $[-A, A + C]$:

$$f(t) = \begin{cases} B & -A \leq t < A \\ 0 & A \leq t < A + C \end{cases}$$

So we can select the period we integrate to be $[-A, A + C]$, so we have

$$\begin{aligned} F_n &= \frac{1}{T} \int_{-A}^{A+C} f(t)\, e^{-jn\omega_0 t}\, dt \\ &= \frac{1}{T} \int_{-A}^{A} B\, e^{-jn\omega_0 t}\, dt + \frac{1}{T} \int_{A}^{A+C} 0\, e^{-jn\omega_0 t} dt \\ &= \frac{B}{T} \left. \frac{e^{-jn\omega_0 t}}{-jn\omega_0} \right|_{t=-A}^{A} \\ &= \frac{B}{-jn\omega_0 T} \left(e^{-jn\omega_0 A} - e^{jn\omega_0 A} \right) \\ &= \frac{B}{jn2\pi} \left(e^{jn\omega_0 A} - e^{-jn\omega_0 A} \right) \\ &= \frac{B}{n\pi} \sin\left(n\omega_0 A\right) = \frac{B}{n\pi} \sin\left(\frac{2\pi n A}{2A + C} \right). \end{aligned}$$

We have one problem. F_0 is not well-defined, since in the expression for F_n, we divide by 0 when $n = 0$, so we have to calculate F_0 separately:

$$F_0 = \frac{1}{T} \int_{t_0}^{t_0+T} f(t)\, dt = \frac{1}{T} \int_{-A}^{A} B\, dt = \frac{2AB}{2A + C}$$

Example 6

For the function $f(t)$ in the previous problem, suppose that $C = 2A = 2$ and that $f(t)$ is the input to an LTI system with frequency response

$$H(\omega) = \begin{cases} 2e^{jn\pi/2} & |\omega| < \pi \\ 0 & \textit{else} \end{cases}$$

Find the output $y(t)$ as a sum of sines and/or cosines.

Solutions

If $C = 2A = 2$, then $f(t)$ is a square wave with some DC offset, the fundamental frequency is $\omega_0 = \frac{\pi}{2}$ and for $n \neq 0$, we have:

$$F_n = \frac{B}{n\pi} \sin\left(\frac{\pi n}{2}\right) \quad \text{and } F_0 = \frac{B}{2}$$

We are sending a sum of scaled exponentials into an LTI system, so we have

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{j\pi nt/2} \rightarrow H(\omega) \rightarrow y(t) = \sum_{n=-\infty}^{\infty} F_n H(\pi n/2) e^{j\pi nt/2}.$$

Thus $y(t)$ is also a sum of scaled exponentials, and

$$F_n\, H(\pi n/2) = \begin{cases} 2F_n e^{jn\pi/2} & |\pi n/2| < \pi \\ 0 & \text{else} \end{cases}$$

$\left|\frac{n\pi}{2}\right| < \pi$ if and only if $n = -1, 0, 1$, i.e. the only terms that "survive" the filter are those indexed by $n = \pm 1, 0$. In particular,

$$F_0 = \frac{B}{2}, \qquad F_1 = \frac{B}{\pi} \sin(\pi/2) = \frac{B}{\pi}, \qquad F_{-1} = \frac{B}{-\pi} \sin(-\pi/2) = \frac{B}{\pi}.$$

and so

$$F_n\, H(\omega_0 n) = \begin{cases} B & n = 0 \\ 2jB/\pi & n = 1 \\ -2jB/\pi & n = -1 \\ 0 & \text{else} \end{cases}$$

where we use the fact $e^{j\pi/2} = j$. Thus

$$\begin{aligned} y(t) &= F_{-1} H(-\pi/2)\, e^{-j\pi t/2} + F_0 H(0) + F_1 H(\pi/2)\, e^{j\pi t/2} \\ &= B + \frac{2jB}{\pi} \left(e^{j\pi t/2} - e^{-j\pi t/2}\right) \\ &= B - \frac{4B}{\pi} \left(\frac{e^{j\pi t/2} - e^{-j\pi t/2}}{2j}\right) = B - \frac{4B}{\pi} \sin(\pi t/2) \end{aligned}$$