UC San Diego J. Connelly

ECE 45 Discussion 5 Notes

Fourier Series Properties

We can represent a periodic function as a sum of sinusoidal components with different coefficients, i.e. a Fourier Series. However, the calculations of the coefficients is often tedious and time-consuming. We can use the properties of the Fourier series to simplify calculations of similar signals. Proving each of these properties is a good exercise.

Fourier Series as an Input to an LTI System

A Fourier series is a sum of sinusoidal components. We know how to analyze LTI systems for sinusoidal inputs, so by linearity, we can determine the output when we have a periodic input. Let $x(t)$ be periodic with fundamental frequency ω_0, then

$$x(t) = \sum_{n=-\infty}^{\infty} X_n \, e^{j\omega_0 nt} \longrightarrow H(\omega) \longrightarrow y(t) = \sum_{n=-\infty}^{\infty} H(n\omega_0) X_n \, e^{j\omega_0 nt}$$

Notice that $y(t)$ is also a periodic function with fundamental frequency ω_0. We can represent $y(t)$ using a Fourier series with coefficients:

$$Y_n = X_n H(n\omega_0)$$

Linearity and Scaling

If we have two periodic functions $f(t)$ and $g(t)$ with fundamental frequency ω_0 and coefficients F_n and G_n respectively.

$$\text{Let } h(t) = a\, f(t) + b\, g(t)$$

$h(t)$ will also be periodic with fundamental frequency ω_0 and coefficients:

$$H_n = a\, F_n + b\, G_n$$

Time Reversal

If we have two periodic functions $f(t)$ and $g(t)$ such that $g(t) = f(-t)$, then $G_n = F_{-n}$.

Time Shifting

If we have two periodic functions $f(t)$ and $g(t)$ such that $g(t) = f(t - t_0)$, then $G_n = F_n \, e^{-j\omega_0 n t_0}$.

Time Scaling

If we have two periodic functions $f(t)$ and $g(t)$ such that $g(t) = f(at)$ and the fundamental frequency of $f(t)$ is ω_0, then $G_n = F_n$, but the fundamental frequency of $g(t)$ is $a\omega_0$.

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Time Derivative

If we have two periodic functions $f(t)$ and $g(t)$ such that $g(t) = \dfrac{df(t)}{dt}$, then $G_n = j\omega_0 n F_n$.

Time Multiplication

If we have three periodic functions $f(t)$, $g(t)$, and $h(t)$ such that $h(t) = f(t)\,g(t)$, the coefficients for $h(t)$ can be determined from the coefficients for $f(t)$ and $g(t)$

$$H_n = \sum_{k=-\infty}^{\infty} F_k\, G_{n-k}$$

Parseval's Theorem

The *average power in a period* of a periodic function can be calculated two ways:

$$P_{avg} = \frac{1}{T}\int_T |f(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |F_n|^2$$

Sometimes the calculations for one method are much simpler than the other.

Real Functions

If $f(t)$ is a real function, then $F_n = F^*_{-n}$. The Fourier series of a real function can be simplified to a sum of sines and cosines.

This follows from the fact that for a real function $f(t) = f^*(t)$.

Imaginary Functions

If $f(t)$ is an imaginary function, then $F_n = -F^*_{-n}$

This follows from the fact that for an imaginary function $f(t) = -f^*(t)$.

Even Functions

If $f(t)$ is an even function, then $F_n = F_{-n}$. The Fourier series of an even function can be simplified to a sum of cosines.

This follows from the fact that for an even function $f(t) = f(-t)$.

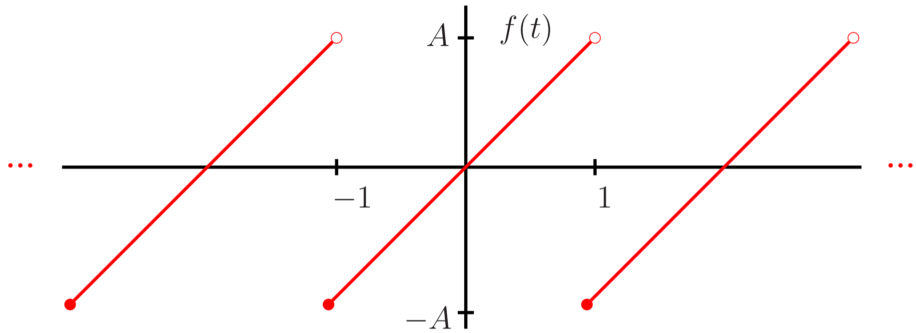
Odd Functions

If $f(t)$ is an odd function, then $F_n = -F_{-n}$. The Fourier series of an odd function can be simplified to a sum of sines.

This follows from the fact that for an odd function $f(t) = -f(-t)$.

Example 1

Write $f(t)$ as a sum of sines and cosines and find the average power in a period, where



Solutions

We note that $f(t)$ has period 2 and in the period $[-1,1)$, $f(t) = At$. So for all $n \neq 0$, we have

$$\begin{aligned}
F_n &= \frac{1}{T}\int_T f(t)e^{-j\omega_0 nt}\,dt = \frac{A}{2}\int_{-1}^{1} te^{-j\pi nt}\,dt \\
&= \frac{A}{-j2\pi n}\left(te^{-j\pi nt}\Big|_{-1}^{1} - \int_{-1}^{1} e^{-j\pi nt}\,dt\right) \\
&= \frac{-A}{j2\pi n}\left(e^{-j\pi n} + e^{j\pi n} + \frac{1}{j\pi nt}\left(e^{-j\pi n} - e^{j\pi n}\right)\right) = \frac{-A}{j\pi n}(-1)^n.
\end{aligned}$$

In the second line, we divide by 0 when $n = 0$, so

$$F_0 = \frac{A}{2}\int_{-1}^{1} t\,e^0\,dt = 0.$$

Thus we have

$$\begin{aligned}
f(t) = \sum_{n=-\infty}^{\infty} F_n e^{j\omega_0 nt} &= \sum_{n=1}^{\infty} \frac{-A}{j\pi n}(-1)^n e^{j\pi nt} - \frac{-A}{j\pi n}(-1)^{-n}e^{-j\pi nt} \\
&= -A\sum_{n=1}^{\infty}(-1)^n\left(\frac{e^{j\pi nt} - e^{-j\pi nt}}{j\pi n}\right) = -2A\sum_{n=1}^{\infty}\frac{(-1)^n}{\pi n}\sin(\pi nt)
\end{aligned}$$

The average power in a period is a straight-forward calculation in the time domain:

$$\frac{1}{2}\int_{-1}^{1} f(t)^2\,dt = \frac{A^2}{2}\int_{-1}^{1} t^2\,dt = \frac{A^2}{3}.$$

By Parseval's Theorem:

$$\frac{A^2}{3} = \sum_{n=-\infty}^{\infty}|F_n|^2 = \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left|\frac{-A}{j\pi n}(-1)^n\right|^2 = \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{A^2}{\pi^2 n^2} = \sum_{n=1}^{\infty}\frac{2A^2}{\pi^2 n^2}$$

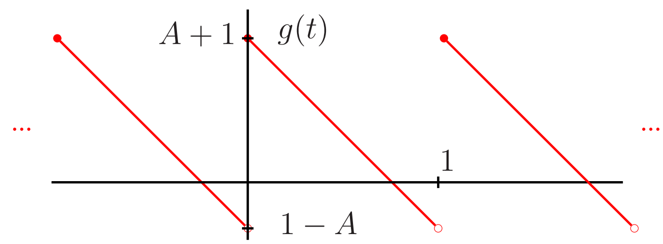
Thus we have

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

which is a beautiful bit of mathematics :)

Example 2

Write $g(t)$ as a sum of sines and cosines and find the Fourier series components G_n, where



Solutions

Note that we have $g(t) = -f(2t-1) + 1$, so

$$g(t) = 1 + 2A \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi n} \sin(\pi n(2t-1)).$$

In order to find G_n in terms of F_n, let's use some intermediate steps:

Let $x(t) = -f(t-1)$, then $x(t)$ is periodic with period 2 and $X_n = -F_n e^{-j\pi n}$.

Let $y(t) = x(2t)$, then $y(t)$ is periodic with period 1 and $Y_n = X_n$.

Then $g(t) = y(t) + 1$, so for $n \neq 0$,

$$G_n = -F_n(-1)^n = \frac{A}{j\pi n}$$

and $G_0 = F_0 + 1 = 1$.

Example 3

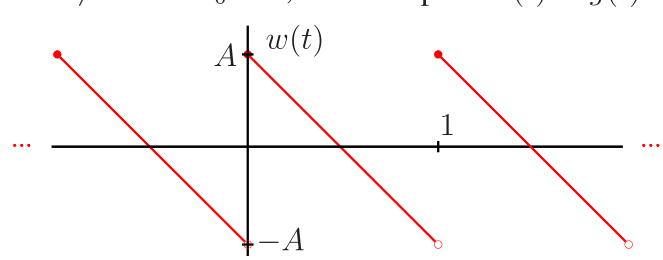
Suppose $g(t)$ is the input to an LTI system with frequency response $H(\omega) = 1$, when $|\omega| > \pi$ and is 0 otherwise. Plot the output $w(t)$.

Solutions

Since $g(t)$ is periodic with period 1, $w(t)$ is also periodic with period 1 and the Fourier series coefficients of $w(t)$ are given by

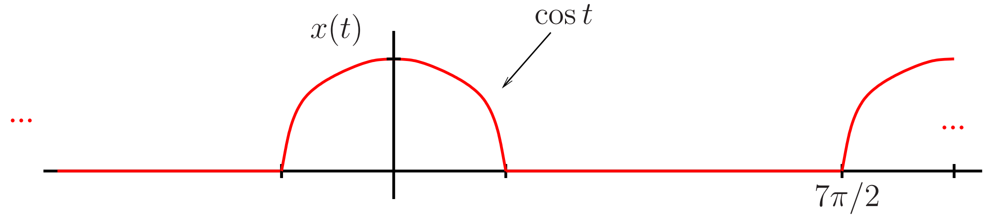
$$W_n = H(\omega_0 n)G_n = H(2\pi n)G_n = \begin{cases} G_n & \text{if } |2\pi n| > \pi \\ 0 & \text{otherwise} \end{cases}$$

So we have $W_n = G_n$ for all $n \neq 0$ and $W_0 = 0$, which implies $w(t) = g(t) - G_0 = g(t) - 1$.



Example 4

Calculate the Fourier series components X_n of $x(t)$. Simplify the expression for X_n to be purely real.



Solutions

$x(t)$ is a periodic function with period $T = 4\pi$ and fundamental frequency $\omega_0 = 1/2$, and

$$x(t) = \begin{cases} \cos t & |t| \le \pi/2 \\ 0 & \pi/2 < t < 7\pi/2 \end{cases}$$

So for all $n \neq \pm 2$, we have

$$\begin{aligned}
TX_n &= \int_T x(t)e^{-j\omega_0 nt} = \int_{-\pi/2}^{\pi/2} \cos(t)e^{-jnt/2}\,dt \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} e^{jt(1-n/2)} + e^{-jt(1+n/2)}\,dt \\
&= \frac{1}{2}\left(\frac{e^{jt(1-n/2)}}{j(1-n/2)} + \frac{e^{-jt(1+n/2)}}{-j(1+n/2)}\right)\Bigg|_{-\pi/2}^{\pi/2} \\
&= \frac{1}{2j}\left(\frac{e^{j\pi/2}e^{-j\pi n/4} - e^{-j\pi/2}e^{j\pi n/4}}{1-n/2} - \frac{e^{-j\pi/2}e^{-j\pi n/4} - e^{j\pi/2}e^{j\pi/4}}{1+n/2}\right) \\
&= \frac{1}{2j}\left(\frac{je^{-j\pi n/4} + je^{j\pi n/4}}{1-n/2} + \frac{je^{-j\pi n/4} + je^{j\pi/4}}{1+n/2}\right) \\
&= \cos(\pi n/4)\left(\frac{1}{1-n/2} + \frac{1}{1+n/2}\right) \\
&= \cos(\pi n/4)\,\frac{2}{4-n^2}
\end{aligned}$$

Since we divide by zero when $n = \pm 2$, we must calculate X_2 and X_{-2} separately, but since $x(t)$ is even, so X_n is also even, so $X_2 = X_{-2}$.

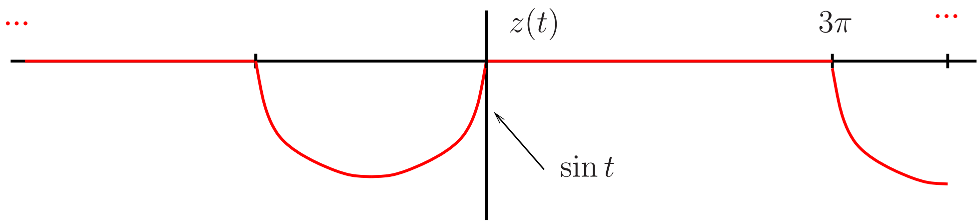
$$\begin{aligned}
TX_2 &= \frac{1}{2}\int_{-\pi/2}^{\pi/2} 1 + e^{-2jt} \\
&= \frac{\pi}{2} + \frac{e^{-2jt}}{-2j}\Bigg|_{-\pi/2}^{\pi/2} \\
&= \frac{\pi}{2} + \frac{e^{j\pi} - e^{-j\pi}}{-2j} = \pi/2.
\end{aligned}$$

Thus for all integers n, we have

$$X_n = \begin{cases} 1/8 & n = \pm 2 \\ \dfrac{\cos(\pi n/4)}{2\pi(4-n^2)} & n \neq \pm 2 \end{cases}$$

Example 5

Express the Fourier series components of $z(t)$ in terms of the Fourier series components of $x(t)$.



Solutions

$z(t)$ is a periodic function with period 4π. Note that

$$z(t) = 1 - x(t + \pi/2).$$

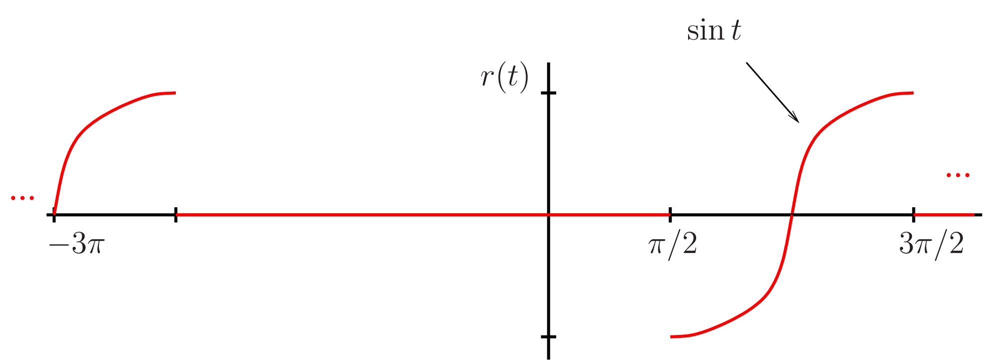
Hence for all $n \neq 0$, we have

$$Z_n = -e^{jn\pi/4} X_n$$

and $Z_0 = 1 - X_0$.

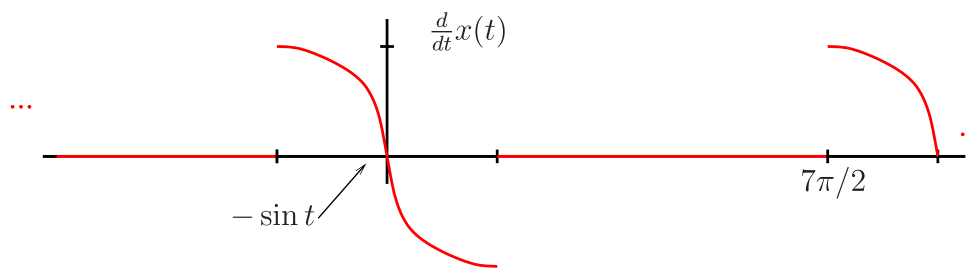
Example 6

Express the Fourier series components of $r(t)$ in terms of the Fourier series components of $x(t)$. What is the DC component of $r(t)$? (i.e. R_0)



Solutions

$r(t)$ is a periodic function with period 4π. Note that taking the derivative of $x(t)$ yields



Then multiplying by -1 and shifting this function to the left by π gives us $r(t)$. Hence

$$r(t) = -\frac{d}{dt} x(t - \pi).$$

Thus by the linearity, time-shifting, and time derivative properties, for all integers n, we have

$$R_n = -\left(j\frac{n}{4}\right) e^{-jn\pi/4} X_n$$

By plugging $n = 0$ into the expression for R_n, we get $R_0 = 0$. Alternatively, recall that

$$R_0 = \frac{1}{T} \int_T r(t)\, dt$$

which is the average value of $r(t)$ in a period T. The average value of $r(t)$ is clearly 0.