ECE 45 Discussion 6 Notes

Fourier Transform

Prior to now, our analysis of linear systems (circuits) has been limited to periodic functions. How can we analyze outputs of LTI systems when the inputs are not periodic? If we could represent an aperiodic function as a sum of sinusoids (as we did with periodic functions using the Fourier series), then we could analyze LTI systems with aperiodic inputs.

We can do exactly this using the *Fourier transform* to represent a function as a continuous sum (i.e. an integral) of sinusoidal components. We can think of an integral as a summation where the distance between the terms approaches zero, so the Fourier transform can be thought of as a Fourier series with infinite period (thus $\omega_0 \rightarrow 0$).

For a function f(t), we can write the Fourier transform of f(t), denoted $F(\omega)$, as:

$$F(\omega) = \mathcal{F}(f(t)) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

We can write f(t) in terms of its Fourier transform by:

$$f(t) = \mathcal{F}^{-1}(F(\omega)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

For periodic functions, the components of the Fourier series, F_n , each represent the contribution from the sinusoid with frequency $n\omega_0$. The components of the Fourier transform are represented as a continuous function, rather than discrete points, i.e. rather than only having contributions from sinusoids at frequencies at multiples of the fundamental frequency, non-periodic functions can have contributions from sinusoids at ALL frequencies. This is akin to having $\omega_0 \rightarrow 0$.

Fourier Transform as an Input to an LTI System:

The Fourier transform is an integral (similar to a sum with infinitesimally small intervals) of sinusoidal components. We can use this property and the fact the system is linear to give us an input output relationship. For any input x(t).

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \, e^{j\omega t} \, d\omega \longrightarrow H(\omega) \longrightarrow y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) \, X(\omega) \, e^{j\omega t} \, d\omega$$

We can represent y(t) as its Fourier Transform:

$$Y(\omega) = X(\omega) H(\omega)$$

Note: $y(t) \neq x(t) h(t) \neq x(t) H(\omega)$, etc. $H(\omega)$ is the Fourier transform of the *impulse response* h(t).

The Fourier transform is a one-to-one mapping, so if we know either the time function or the frequency function, we know the other as well. It is often convenient or useful to go back and forth between the time and frequency representations of functions.

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Other applications

Our main use of the FT right now is for LTI system analysis, but this extends to many different applications. If we are transmitting or broadcasting data, we may care about the *bandwidth* that we use. By taking the Fourier transform of a signal, we can see what range of frequencies contribute to our signal.

Different materials respond differently to different frequency signals (i.e. water essentially acts as a LPF, blocking high frequencies and allowing low). By knowing what frequencies our signal spans, we can determine if our signal will deteriorate.

Example 1

For real numbers $t_0 \ge 0$ and a > 0, determine the Fourier transforms (Frequency domain representation) of f(t) and g(t), where

$$f(t) = \begin{cases} 0 & \text{for } t < t_0 \\ e^{-a(t-t_0)} & \text{for } t \ge t_0 \end{cases} \text{ and } g(t) = \begin{cases} 0 & \text{for } t > t_0 \\ e^{a(t-t_0)} & \text{for } t \le t_0 \end{cases}$$

Solutions

Both are fairly straight-forward calculations from the definition of the Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \int_{t_0}^{\infty} e^{-a(t-t_0)} e^{-j\omega t}dt$$
$$= e^{at_0} \int_{t_0}^{\infty} e^{-t(a+j\omega)}dt = e^{at_0} \left. \frac{e^{-t(a+j\omega)}}{-(a+j\omega)} \right|_{t=t_0}^{\infty}$$
$$= e^{at_0} \left. \frac{e^{-t_0(a+j\omega)} - 0}{(a+j\omega)} \right|_{t=t_0}^{\infty}$$

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt = \int_{-\infty}^{t_0} e^{a(t-t_0)} e^{-j\omega t}dt$$
$$= e^{-at_0} \int_{-\infty}^{t_0} e^{t(a-j\omega)}dt = e^{-at_0} \frac{e^{t(a-j\omega)}}{(a-j\omega)} \Big|_{t=-\infty}^{t_0}$$
$$= e^{-at_0} \frac{e^{t_0(a-j\omega)} - 0}{(a-j\omega)} = \frac{e^{-j\omega t_0}}{a-j\omega}$$

Note that if $t_0 = 0$, then f(t) = g(-t) and $F(\omega) = G(-\omega)$. We will generalize this property later on.

Example 2

Determine the output to an LTI system $H(\omega)$ when the input is $x(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-at} & \text{for } t \ge 0 \end{cases}$ and the impulse response of $H(\omega)$ is $h(t) = \begin{cases} 0 & \text{for } t > 0 \\ e^{at} & \text{for } t \le 0 \end{cases}$.

Solutions

From the previous problem x(t) is f(t) with $t_0 = 0$ and h(t) is g(t) with $t_0 = 0$, so we have:

$$\begin{split} X(\omega) &= \frac{1}{a + j\omega} \ \text{and} \ H(\omega) = \frac{1}{a - j\omega}. \\ x(t) \ \to \ H(\omega) \ \to \ y(t) \\ \Rightarrow Y(\omega) &= X(\omega) \ H(\omega) = \frac{1}{a + j\omega} \ \frac{1}{a - j\omega}. \end{split}$$

We could try to compute the inverse Fourier transform of $Y(\omega)$ at this point, but the integral will be messy... instead let's try to get $Y(\omega)$ in a form we know the inverse Fourier transform of. Using partial fractions, we have:

$$Y(\omega) = \frac{A}{a+j\omega} + \frac{B}{a-j\omega}$$
$$= \frac{A(a-j\omega) + (B(a+j\omega))}{(a+j\omega)(a-j\omega)}$$
$$\Rightarrow 1 = A(a-j\omega) + B(a+j\omega)$$
$$\Rightarrow 1/a = (A+B) \text{ and } 0 = B - A \Rightarrow A = B = \frac{1}{2a}$$

and so we have

$$Y(\omega) = \frac{1}{2a} \left(\frac{1}{a+j\omega} + \frac{1}{a-j\omega} \right) = \frac{1}{2a} \left(X(\omega) + H(\omega) \right)$$

Taking the Fourier transform is a linear operation so:

$$y(t) = \frac{1}{2a} \left(x(t) + h(t) \right) = \frac{1}{2a} \begin{cases} e^{at} & \text{for } t \le 0 \\ e^{-at} & \text{for } t \ge 0 \end{cases} = \frac{1}{2a} e^{-a|t|}$$

Example 3

If $x(t) = e^{-2|t|} \cos(2t)$ is the input to an LTI system and $y(t) = 0.5 e^{-2|t-1|} \cos(2t-2)$ is the output, what is the frequency response $H(\omega)$ of the system?

Solutions

Recall $Y(\omega) = X(\omega)H(\omega)$, so if we can calculate $Y(\omega)$ and $X(\omega)$, we can find $H(\omega)$. However, this is a potentially messy calculation, so if we can write $Y(\omega)$ in terms of $X(\omega)$, we may be able to find $H(\omega)$ without much calculation.

Notice y(t) = 0.5 x(t-1). We can write y(t) in terms of its Fourier transform and do some manipulation to find $Y(\omega)$ in terms of $X(\omega)$.

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{j\omega t} d\omega = y(t) = 0.5 x(t-1) = 0.5 \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega(t-1)} d\omega\right)$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(0.5 X(\omega) e^{-j\omega}\right) e^{j\omega t} d\omega$$

Since the Fourier transform is a one-to-one mapping, this implies $Y(\omega) = 0.5 X(\omega) e^{-j\omega}$, and so

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = 0.5 e^{-j\omega}$$

Example 4

Determine the Fourier transform of the function g(t) = f(t) x(t) where

$$f(t) = \begin{cases} 0 & \text{for } t < 0\\ e^{-at} & \text{for } t \ge 0 \end{cases} \quad \text{and} \ x(t) = t$$

Solutions

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt = \int_{0}^{\infty} t e^{-at} e^{-j\omega t} dt = \int_{0}^{\infty} t e^{-t(a+j\omega)} dt$$

Need to use integration by parts, so let

$$\begin{split} u(t) &= t \quad \rightarrow \ \frac{du(t)}{dt} = 1 \\ \frac{dv(t)}{dt} &= e^{-t(a+j\omega)} \quad \rightarrow \ v(t) = \frac{e^{-t(a+j\omega)}}{-(a+j\omega)} \end{split}$$

$$G(\omega) = \int_0^\infty t \, e^{-t(a+j\omega)} \, dt = \int u(t) \, \frac{dv(t)}{dt} \, dt = u(t) \, v(t) \Big|_0^\infty - \int_0^\infty v(t) \, \frac{du(t)}{dt} \, dt$$
$$= t \, \frac{e^{-t(a+j\omega)}}{-(a+j\omega)} \Big|_0^\infty + \frac{1}{a+j\omega} \int_0^\infty e^{-t(a+j\omega)} \, dt = \frac{e^{-t(a+j\omega)}}{(a+j\omega)^2} \Big|_0^\infty = \frac{1}{(a+j\omega)^2}.$$

Note:
$$\lim_{t \to \infty} t e^{-t(a+j\omega)} = t e^{-ta} e^{-tj\omega} = \left(\lim_{t \to \infty} t e^{-at}\right) \left(\lim_{t \to \infty} e^{-tj\omega}\right)$$
$$= 0 \left(\lim_{t \to \infty} e^{-tj\omega}\right) = 0.$$