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ECE 45 Discussion 7 Notes

Fourier Transform

The Fourier transform is a way of representing a function in terms of its frequency components. We can think of it as a Fourier series with an infinite period (i.e. it never repeats itself), so it can have frequency contributions from any frequency.

$$f(t) = \mathcal{F}^{-1}(F(\omega)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega \quad \text{and} \quad F(\omega) = \mathcal{F}(f(t)) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt$$

Fourier Transform as an Input to an LTI System

Since a Fourier transform is an integral (sum with infinitesimally small intervals) of sinusoidal components, we know how to analyze the output of a system with a periodic function as its input. For any input $x(t)$ and any LTI system $H(\omega)$, we have

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega \longrightarrow H(\omega) \longrightarrow y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega) X(\omega)\, e^{j\omega t} d\omega$$

We can represent the output $y(t)$ as its Fourier Transform: $Y(\omega) = X(\omega)H(\omega)$

Linearity of the FT

if $f_1(t) \longleftrightarrow F_1(\omega)$ and $f_2(t) \longleftrightarrow F_2(\omega)$

$$\text{then } a\, f_1(t) + b\, f_2(t) \longleftrightarrow a\, F_1(\omega) + b\, F_2(\omega)$$

Other Properties

If $F(\omega)$ is the Fourier transform of $f(t)$, then

$f(t - t_0)$	$\longleftrightarrow$	$F(\omega)\, e^{-j\omega t_0}$	**Time Shifting**
$f(t)\, e^{j\omega_0 t}$	$\longleftrightarrow$	$F(\omega - \omega_0)$	**Frequency Shifting**
$f^*(t)$	$\longleftrightarrow$	$F^*(-\omega)$	**Conjugation**
$f(c\, t)$	$\longleftrightarrow$	$\frac{1}{\lvert c \rvert} F\left(\frac{\omega}{c}\right)$	**Time Scaling**
$\frac{d}{dt} f(t)$	$\longleftrightarrow$	$j\omega F(\omega)$	**Time Derivative**
$-jt f(t)$	$\longleftrightarrow$	$\frac{d}{d\omega} F(\omega)$	**Frequency Derivative**
$F(t)$	$\longleftrightarrow$	$2\pi\, f(-\omega)$	**Duality/Symmetry Property**

Note: The Fourier Transform is NOT multiplicative: $f(t)g(t) \not\longleftrightarrow F(\omega)G(\omega)$

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These properties hold for any Fourier transform pair and are very useful to use as they allow us to determine Fourier transforms of similar functions without having to do a lot of calculations. For the most part, if we can write a function $g(t)$ in terms of a function we know the Fourier transform of, $f(t)$, then we can easily determine the Fourier transform of $g(t)$.

It is a very useful exercise to derive the properties using the definition of the Fourier Transform:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega, \text{ where } F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

Parseval's Theorem

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega$$

Unit Step Function (Heaviside step function)

Useful for representing a signal "turning on" or starting at a certain value.

$$u(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 & \text{for} \quad t \geq 0 \end{cases}$$

By adding and subtracting time-shifted unit step functions, we can represent "windowed" functions. e.g.

$$f(t)\left(u(t) - u(t-5)\right) = \begin{cases} f(t) & \text{for} \quad 0 \leq t < 5 \\ 0 & \quad\text{else} \end{cases}$$

Example 1

Find the Fourier transform of $f(t) = t\left(u(t+1) - u(t-1)\right)$

Solutions

$f(t) = t$, for $t \in [-1, 1)$, and $f(t) = 0$, otherwise, so

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt = \int_{-1}^{1} te^{-j\omega t}\,dt = \frac{-1}{j\omega}\left(te^{-j\omega t}\Big|_{-1}^{1} - \int_{-1}^{1} e^{-j\omega t}\,dt\right) \\ &= \frac{-1}{j\omega}\left(e^{-j\omega} + e^{j\omega} + \frac{e^{-j\omega} - e^{j\omega}}{j\omega}\right) = \frac{-1}{j\omega}\left(2\cos(\omega) - \frac{2\sin\omega}{\omega}\right) = \frac{2}{j\omega}\left(\text{sinc}\,(\omega) - \cos(\omega)\right) \end{aligned}$$

Example 2

For real numbers $A, B > 0$, find the Fourier transform of $g(t) = At\left(u(t+B) - u(t-B)\right)$

Solutions

$g(t) = At$, for $t \in [-B, B)$, and $g(t) = 0$, otherwise.
$f(t/B) = t/B$, for $t \in [-B, B)$, and $f(t/B) = 0$, otherwise, so we have

$$g(t) = ABf(t/B)$$

so by the amplitude and time-scaling properties,

$$G(\omega) = AB^2F(B\omega) = \frac{2AB}{j\omega}\left(\text{sinc}\,(B\omega) - \cos(B\omega)\right)$$

Example 3

Find the inverse Fourier transform of $X(\omega)$, where $X(\omega) = \pi$, for $|\omega| \leq 1$ and $X(\omega) = 0$ otherwise.

Solutions

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega = \frac{1}{2}\int_{-1}^{1} e^{j\omega t}\,d\omega = \frac{1}{2}\,\frac{e^{jt} - e^{-jt}}{j\omega} = \frac{\sin t}{t} = \text{sinc}\,(t)$$

Example 4

Evaluate the integral $\int_0^{\infty} \frac{\sin^2(t)}{t^2}\,dt$.

Solutions

The function $\frac{\sin^2(t)}{t^2}$ is even, so

$$\int_0^{\infty} \frac{\sin^2(t)}{t^2}\,dt = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin^2(t)}{t^2}\,dt = \int_{-\infty}^{\infty} |\text{sinc}\,(t)\,|^2\,dt$$

which we can calculate using Parseval's Theorem

$$\int_{-\infty}^{\infty} |\text{sinc}\,(t)\,|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\mathcal{F}\left(\text{sinc}\,(t)\right)\right|^2\,d\omega = \frac{1}{2\pi}\int_{-1}^{1} \pi^2\,d\omega = \pi$$

$$\text{Thus } \int_0^{\infty} \frac{\sin^2(t)}{t^2}\,dt = \frac{\pi}{2}$$

Example 5

Let A and B be positive real numbers. Find the Fourier transform of $z(t) = e^{-At}\sin(Bt)\,u(t)$.

Solutions

$$
\begin{aligned}
Z(\omega) &= \int_{-\infty}^{\infty} z(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-At}\sin(Bt)e^{-j\omega t}\,dt = \frac{1}{2j}\int_{0}^{\infty} e^{-At}(e^{jBt} - e^{-jBt})e^{-j\omega t}\,dt \\
&= \frac{1}{2j}\int_{0}^{\infty} e^{-t(A-jB+j\omega)} - e^{-t(A+jB+j\omega)}\,dt = \frac{1}{2j}\left(\frac{1}{A+j(\omega-B)} - \frac{1}{A+j(\omega+B)}\right) \\
&= \frac{1}{2j}\,\frac{(A+j(\omega+B)) - (A+(j\omega-B))}{(A+j(\omega-B))(A+j(\omega+B))} = \frac{B}{A^2+2Aj\omega-\omega^2+B^2} = \frac{B}{B^2+(A+j\omega)^2}
\end{aligned}
$$

Example 6

Let A and B be positive real numbers. Find the Fourier transform of $z(t) = te^{-At}\sin(Bt)\,u(t)$.

Solutions

We have $z(t) = ty(t) = -jt\,(jy(t))$, so by the frequency derivative property

$$
Z(\omega) = \frac{d}{d\omega}\mathcal{F}(jy(t)) = \frac{d}{d\omega}(jY(\omega)) = j\frac{-2jB(A+j\omega)}{(B^2+(A+j\omega)^2)^2} = \frac{2B(A+j\omega)}{(B^2+(A+j\omega)^2)^2}
$$